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A QUADRATIC PERFORMANCE INDEX FOR A VTOL AIRCRAFT PREFILTER MODEL REFERENCE ATTITUDE CONTROL SYSTEM

by Terrence D. Gossett and Lloyd D. Corliss

Ames Research Center

and

U. S. Army Air Mobility R&D Laboratory

Moffett Field, Calif. 94035

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16. Abstract

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SYMBOLS

B	model control matrix
B_i	transfer function poles (fig. 8)
b	intercept of log-log plots
b_{ij}	elements of B
C_i	transfer function zeros (fig. 8)
D_1	total outer-loop gain
D_1	total outer-loop gain (second-order example)
D_n	transfer function denominator coefficients
F	total system matrix incorporating system input (nonhomogeneous)
F'	total system matrix (nonhomogeneous)
$\overline{F}$	total system matrix (homogeneous)
F_f	unaugmented follower matrix
f_{ij}	elements of F_f
G	control matrix (incorporating system input)
G'	control matrix
G_f	follower control matrix
g	acceleration due to gravity
g_{ij}	elements of G_f
H	error matrix
h	time increment, sec
I_x,I_y,I_z	moments of inertia
I_{xz}	product of inertia
K	system gain matrix (row)

K^*	optimum system gain matrix (row)
K_m, K_A, K_B, K_C	partitioned portions of K
k	dc forward loop gain term
$k_\delta, k_{\dot{\phi}_m}, k_{\phi_m}$	model feedforward gain terms
$\left.\begin{matrix} k_{\dot{\delta}_X}, k_{\delta_X}, k_{\dot{\delta}_1}, \\ k_{\delta_1}, k_{\dot{\phi}_A}, k_{\phi_A} \end{matrix}\right\}$	follower feedback gain terms for detailed system (see appendix B)
k_0, k_1, k_2, k_3	follower feedback gain terms for representative system
L_p/I_X	angular acceleration about roll axis due to roll rate, 1/sec
$L_{\bar{r}}/I_X$	angular acceleration about roll axis due to yaw rate
L_v/I_X	angular acceleration about roll axis due to sideslip velocity
L_δ/I_X	angular acceleration about roll axis due to a generalized input (control or gust)
M	model matrix
m	slope of log-log plots
m_{ij}	elements of M
N_n	transfer function numerator coefficients
$N(s)$	transfer function numerator
N_p/I_Z	angular acceleration about yaw axis due to roll rate
$N_{\bar{r}}/I_Z$	angular acceleration about yaw axis due to yaw rate
N_v/I_Z	angular acceleration about yaw axis due to sideslip velocity
N_δ/I_Z	angular acceleration about yaw axis due to a generalized input (control or gust)
P	steady-state Riccati matrix
p_{ij}	elements of P
Q	QPI state weighting matrix
q_{ii}	elements of Q

R	QPI control weighting matrix
r	element of R
$\bar{r}$	yaw rate
s	Laplacian operator
U	system control vector
U^*	optimum system control vector
U_o	trim forward speed
u	element of U
V	quadratic performance index (QPI)
V_{min}	minimized QPI
v	sideslip velocity
X	total system state vector incorporating system input
X'	total system state vector (nonhomogeneous; see appendix B)
X_f	follower state vector
X_m	model state vector
X_o	total system state vector initial conditions
Y	error vector
Y_p/m	linear acceleration along Y axis due to roll rate
$Y_{\bar{r}}/m$	linear acceleration along Y axis due to yaw rate
Y_v/m	linear acceleration along Y axis due to sideslip velocity
Y_δ/m	linear acceleration along Y axis due to a generalized input (control or gust)
δ	control displacement, rad
ϵ	error
$\zeta_{( )}$	damping ratio of subscripted system, 1/sec

τ_i	weighting constants
ϕ	roll angle, rad
$\omega_{(\ )}$	natural frequency of subscripted system, rad/sec
QPI	$\int(X^T H^T Q H X + U^T R U)dt$

Subscripts

A	aircraft
a	actuator
c	composite command to the follower (appendix B)
f	follower
i	arithmetic subscript, i = 1, 2, 3, . . .
m	model
n	arithmetic subscript, n = 1, 2, 3, . . .
$\imath$	bending mode

Superscripts

T	transpose
-1	inverse

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Terrence D. Gossett* and Lloyd D. Corliss*

Ames Research Center

SUMMARY

Several performance criteria methods for optimizing a prefilter model reference control system are reviewed, and the quadratic performance index is studied in detail. The prefilter model reference system consisted of a second-order model and a fourth-order follower. The form of this follower system was typical of that required for a VTOL aircraft in hover (e.g., the X–14B). For the VTOL problem, the quadratic performance index was found to be the most versatile and comprehensive criterion. A system designer can iterate weighting matrices for various system states until the desired system response is obtained; that is, if the response is too oscillatory, the elements of the weighting matrices can be changed to produce a less oscillatory response. A major advantage of the quadratic performance index is that it can be minimized through the use of readily available digital computer programs. The solution of these programs provides a set of optimal gains for a given system configuration. Constraints on the use of the quadratic performance index are that the index must be expressed in terms of states and the control vector, all states must be observable, and only linear system formulations are acceptable.

The shortcomings of applying the quadratic performance index are basically twofold: first, the process for selecting the weighting matrices is not well defined; and second, the parametric relationships of the weighting matrices to the resulting optimum gains are largely unknown. This report presents a simple graphical method for approximating the optimal gains. By this method asymptotes are generated for the inner-loop feedback gains in terms of the total outer-loop gain over a wide range of weighting. The asymptotes approximate the loop gains for the inner feedback and feedforward terms.

The follower total outer-loop gain was found to be a useful reference parameter in constructing system asymptotes. The total outer-loop gain establishes the system "stiffness." Analysis of the canonical form of the state equation and the Riccati equation for a single input single control system indicated that the follower total outer-loop gain could be analytically determined for type-zero or non-type-zero systems. However, only constant numerator, non-type-zero systems are studied in this report. The total outer-loop gain for these systems can be expressed in terms of the attitude weighting element of the Q matrix, the control weighting element of the R matrix, and the element from the control matrix, G.

*U.S. Army Air Mobility R&D Laboratory, Moffett Field, California

INTRODUCTION

Historically, response feedback systems represent the conventional approach to augmentation of the inherent stability of aircraft. However, the expansion of aircraft flight envelopes and variations in basic aircraft dynamic response due to changes in aircraft mass, inertia, and geometry have indicated a need for a more versatile class of flight control systems. The approach taken in this report is to consider a class of systems (i.e., model reference) that has inherent adaptability advantages and then to concentrate on the optimization of a fixed operating point of the system (the example used in this report is a VTOL aircraft in hover). The underlying intent is that one can then achieve adaptability by either discretely or continuously changing the system to comply with the chosen fixed operating points. The type of system felt to best fulfill the inherent adaptability requirement and also lend itself to the research investigation application is a model reference system of the prefilter model following type. This particular form has also been called the prefilter model, the model controlled system, the model reference adaptive system, or the model-in-the-system approach (refs. 1 to 5).

The selection of an optimum model reference system for an aircraft application may be considered in two parts (as shown in fig. 1), both of which require the implementation of a performance criterion. The first portion of the problem involves the selection of a desirable model. This involves a man-machine interface for which the criterion of pilot opinion is most often used. This particular criterion is subjective but, nonetheless, seems well suited and, to date, the best available for such an interface. Several studies have been conducted utilizing pilot opinion to determine desirable hovering VTOL aircraft models in roll, pitch, and yaw (refs. 6 and 7). The roll model used in this study was one selected from these previous investigations (appendix A).

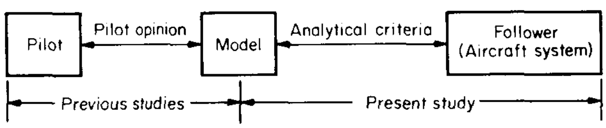

Figure 1.– Generalized model-follower representation.

The second portion of the problem, which is the focal point of this study, involves the optimization of the model-follower combination (fig. 1). For this link, which consists of a machine-machine interface, many performance criteria are available ranging from nonanalytic and rather subjective criteria, such as the comparison of time histories, to more analytical approaches, such as the quadratic performance index (QPI). Various criteria are discussed in appendix A. Nonanalytical criteria have the pitfall of not yielding a standard reference; that is, their "values" may vary from observer to observer. To develop a mean standard reference for such criteria requires the use of statistical methods. While a nonanalytical criterion, such as a pilot opinion, seems appropriate for the pilot-model link, it is deemed that an analytical criterion can and should be used for the model-follower link. This study, then, is concerned with the latter consideration, that is, the determination and interpretation of a satisfactory analytical performance criterion for the model-follower optimization.

Having specified a prefilter model reference system through the optimization of these two links, the final test for the acceptability of the system rests with the pilot. Hence, the optimization of the follower through analytical means should not unduly influence the pilot's appraisal of the system. In short, the pilots assessment of the overall system should hopefully be consistent with his assessment of the model alone.

For the model-follower optimization, several possible performance criteria were reviewed (refs. 8–13). These criteria included Integral Absolute Error, Integral Square Error, Integral Time Absolute Error, Butterworth, and the Quadratic Performance Index (QPI). The QPI, which may be minimized by solving the matrix Riccati equation, was found to be the most desirable criterion (refs. 5, 6, and 9). (See appendix A.) While reference 5 established a good basis for the application of the QPI, certain shortcomings of this technique were cited: (1) The process for selecting the QPI weighting matrices is not well defined. (2) Parametric relationships of these weighting matrices to the optimum gains produced are largely unknown.

Alleviation of the latter limitation would enhance the use of QPI in practice on a synthesis problem. If parametric relationships are known, the designer could make an "on the spot" alteration of the system gains in an *optimal manner*. Presently, to achieve such an optimal alteration of gains for a given system requires a complete iteration of the Riccati equation solution. Establishment of a set of parametric relationships would in turn provide further insight on the first shortcoming cited above, that is, the selection of the weighting matrices. Hence, for a specific class of system, the main objective of this report is to relate analytically and graphically the effects of variations in the QPI weighting elements on the optimum gains they produce. While the relationships developed in this report are with reference to the X–14B VTOL model reference system, the technique need in no way be restricted to such a system.

GUIDELINES FOR INVESTIGATION

For the purposes of this study it will be assumed that the X–14B system will consist of a three-axis (roll, pitch, and yaw) prefilter model reference system that will utilize a general purpose digital computer and an autopilot for which the roll axis is shown on figure 2. The guidelines for investigation of performance criteria that follow were definitely influenced by this assumption. The general guidelines and procedures that will be observed in this study of performance criteria are outlined below:

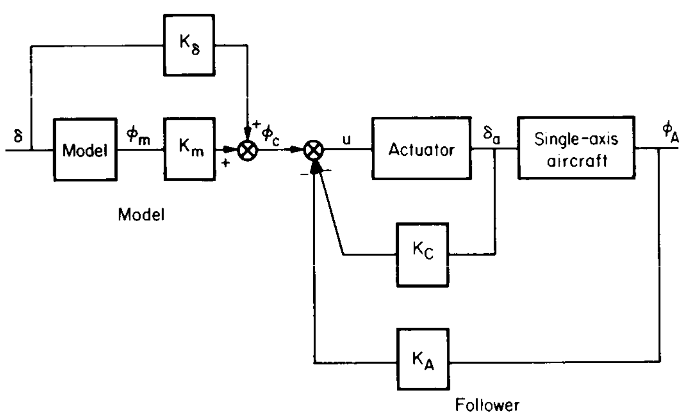

Figure 2.– Simplified roll axis model-follower diagram.

The form of the linearized lateral equations of motion characterizing hovering vehicles that will be used in this report is taken from reference 14.

$$\begin{bmatrix} \left(s - \frac{Y_v}{m}\right) & \left(-\frac{Y_p}{m} s - g\right) & \left(U_o - \frac{Y_{\bar{r}}}{m}\right) \\ \left(-\frac{L_v}{I_x}\right) & \left(s^2 - \frac{L_p}{I_x} s\right) & -\left(\frac{I_{xz}}{I_x} s + \frac{L_{\bar{r}}}{I_x}\right) \\ \left(-\frac{N_v}{I_z}\right) & -\left(\frac{I_{xz}}{I_z} s + \frac{N_p}{I_z}\right) s & \left(s - \frac{N_{\bar{r}}}{I_z}\right) \end{bmatrix} \begin{bmatrix} v \\ \phi_A \\ \bar{r} \end{bmatrix} = \begin{bmatrix} \frac{Y_{\delta_a}}{m} \\ \frac{L_{\delta_a}}{m} \\ \frac{N_{\delta_a}}{I_z} \end{bmatrix} \delta_a$$

For hover, $U_o = 0$ and also for the simplifying, but generally valid, assumption that $I_{xz} = N_p/I_z = Y_{\bar{r}}/m = L_{\bar{r}}/I_x = L_v/I_x = 0$, these equations reduce to:

$$\begin{bmatrix} \left(s - \frac{Y_v}{m}\right) & \left(-\frac{Y_p}{m} s - g\right) & 0 \\ 0 & s\left(s - \frac{L_p}{I_x}\right) & 0 \\ -\frac{N_v}{I_z} & 0 & \left(s - \frac{N_{\bar{r}}}{I_z}\right) \end{bmatrix} \begin{bmatrix} v \\ \phi_A \\ \bar{r} \end{bmatrix} = \begin{bmatrix} \frac{Y_{\delta_a}}{m} \\ \frac{L_{\delta_a}}{m} \\ \frac{N_{\delta_a}}{I_z} \end{bmatrix} \delta_a$$

This report uses a single-axis roll transfer function for hover; hence it is necessary to find the transfer function for ϕ_A/δ_a. Therefore, Y_{δ_a}/m, L_{δ_a}/m, and N_{δ_a}/I_z will be considered to be accelerations from roll input only. In hover, the roll control generally causes only small side forces or yawing moments; hence Y_{δ_a}/m and N_{δ_a}/I_z will be set to zero. Then from the above equation, ϕ_A/δ_a is found to be:

$$\frac{\phi_A}{\delta_a} = \frac{\left(s - \frac{Y_v}{m}\right)\left(\frac{L_{\delta_a}}{I_x}\right)\left(s - \frac{N_{\bar{r}}}{I_z}\right)}{\left(s - \frac{Y_v}{m}\right)\left(s^2 - \frac{L_p}{I_x} s\right)\left(s - \frac{N_{\bar{r}}}{I_z}\right)} = \frac{\frac{L_{\delta_a}}{I_x}}{s^2 - \frac{L_p}{I_x} s}$$

The roll acceleration due to reaction nozzle input, L_{δ_a}/I_x, on the modified X–14 is estimated to be 4.585°/sec²/deg, and the measured value for L_p/I_x for the X–14 was –0.45 sec^{-1}. Thus the transfer function that will be used for ϕ_A/δ_a for hover is

$$\frac{\phi_A}{\delta_a} = \frac{4.585}{s(s + 0.45)}$$

In hover, the X–14 obtains its roll moment from reaction nozzles that are driven by a 10–cps actuator ($\zeta_a = 0.5$) with a dc gain of 1.0°/deg. Hence the actuator transfer function is

$$\frac{\delta_a}{u} = \frac{(62.8)^2}{s^2 + 2(0.5)(62.8)s + (62.8)^2}$$

Therefore, the plant representation of the basic aircraft is

$$\frac{\phi_A}{u} = \frac{(62.8)^2(4.585)}{[s^2 + 2(0.5)(62.8)s + (62.8)^2]s(s + 0.45)}$$

$$= \frac{18,100}{s(s^3 + 63.3s^2 + 3978s + 1775)}$$

Three other guidelines were established. Only the prefilter model reference system will be considered. The model in the examples will be second order with values of $\omega_m = 2$ rad/sec and $\zeta_m = 0.7$ (which corresponds to a satisfactory hover handling qualities model as reported in ref. 7). The error to be minimized will be the difference between the corresponding responses of the model and the follower (e.g., the difference between the model and the follower attitudes; see fig. 2).

ANALYTICAL INVESTIGATIONS

Simply stated, the model following concept requires that the response of the follower match the response of the model, that is, in figure 2, a matching of ϕ_A to ϕ_m. This implies then that any difference in the responses of the follower and the model constitutes an error signal. It is this error signal and its minimization that forms the basis of the various performance criteria discussed herein.

One of the major objectives of this research was to better understand the effect of performance criteria on the system optimums that they produce. The search for a proper criterion was evolutionary in that the simpler criteria were explored first, deficiencies and merits were noted, and then more complex criteria were considered.

QPI Criteria

The quadratic performance index (QPI) was felt to be the most comprehensive and satisfactory criterion (refs. 5, 6, 9, and 15; see appendix A.) It can be minimized by solving the matrix Riccati equation with the aid of a digital computer and the programming package Automatic Synthesis Program (ASP) (ref. 9) or the more recently developed Fortran IV version of Automatic Synthesis Program (FASP) (ref. 16). The expressions used in this system optimization are given below; the terms are defined in the Symbols. These relationships are explained and developed in appendix B.

$$V = \int (X^T H^T Q H X + U^T R U) dt \qquad \text{Quadratic performance index} \qquad (1)$$

$$\dot{P} = 0 = P G R^{-1} G^T P - P F - F^T P - H^T Q H \qquad \text{Steady-state Riccati equation} \qquad (2)$$

$$U^* = -R^{-1} G^T P X = K^* X \qquad \text{Optimum control} \qquad (3)$$

$$V_{min} = \int X^T (H^T Q H + P G R^{-1} G^T P) X \, dt \qquad \text{Minimized QPI} \qquad (4)$$

$$\dot{X} = FX + GU \qquad \text{System equation} \qquad (5)$$

The system given by equation (5) may be optimized, that is, the QPI may be minimized, through the solution of the Riccati equation. The matrix P is the solution to the matrix Riccati equation and is used in generating the optimal control U*.

It was recognized that various aspects of the application of the QPI are not easily understood; hence much of this investigation was directed toward the problem of gaining a better understanding of the optimum gains produced by this technique. This problem centers primarily on understanding the import of the weighting matrices Q and R and on developing a way of relating these Riccati optimums to other existing techniques. It is felt that the most effective way to explain the application of the QPI criterion is to present an example.

Example for QPI

This section will present the mathematical formulation of the optimal control problem described by equations (1) through (5) for a symbolic example. Then an analytical expression will be developed through the use of the matrix Riccati equation for one of the optimal gain terms.

The example for this study is a prefilter model following system with the second-order model and fourth-order follower of the form shown in figure 3. Figure 3 is a generalized form of the system to be used on the X–14B. The QPI weighting will be on the cost of control and on either the matching of attitudes or of attitudes and rates. The state equation for this example and the definition of the various matrices used in the QPI optimization procedure are given below. Further detail of the state space modeling and solution techniques appears in appendix B.

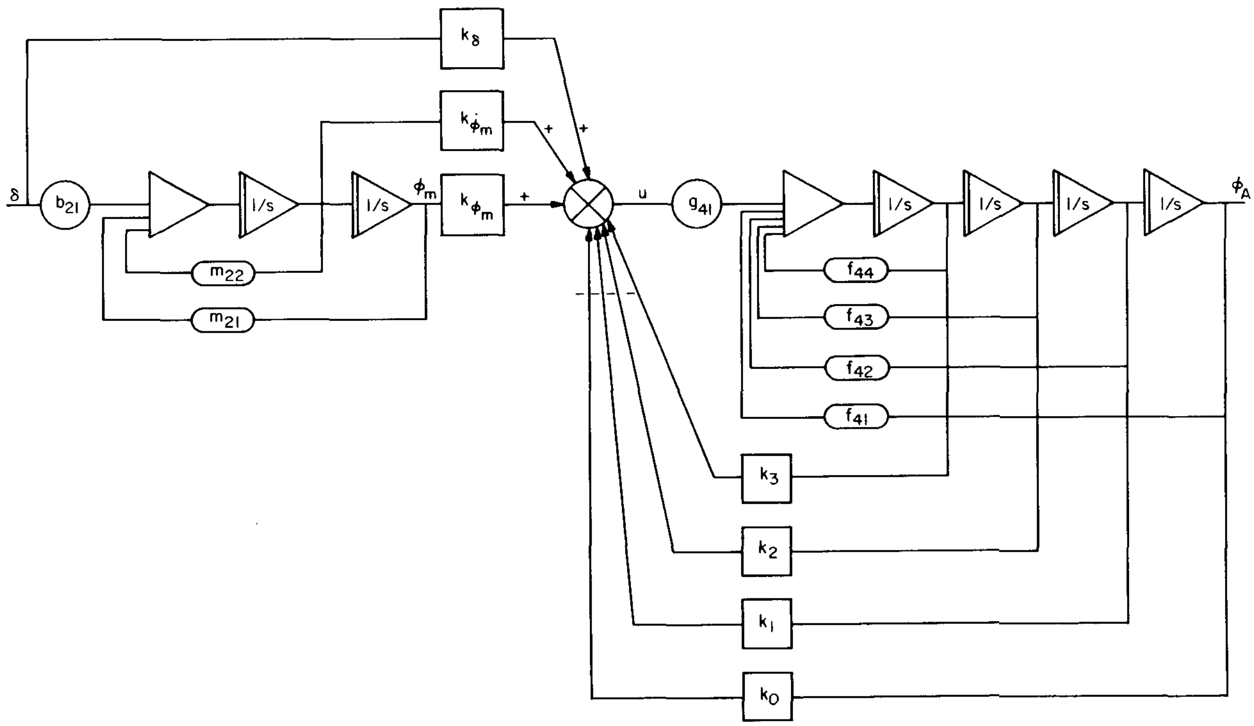

Figure 3.– Detailed roll axis model-follower diagram.

The nonhomogeneous state equation, that is,

$$\dot{X} = FX + GU \tag{5}$$

when expanded for the generalized fourth-order follower system of figure 3, yields

$$\frac{d}{dt}\begin{bmatrix} \phi_A \\ \dot{\phi}_A \\ \ddot{\phi}_A \\ \dddot{\phi}_A \\ \phi_m \\ \dot{\phi}_m \\ \delta \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ f_{41} & f_{42} & f_{43} & f_{44} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & m_{21} & m_{22} & b_{21} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \phi_A \\ \dot{\phi}_A \\ \ddot{\phi}_A \\ \dddot{\phi}_A \\ \phi_m \\ \dot{\phi}_m \\ \delta \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ g_{41} \\ 0 \\ 0 \\ 0 \end{bmatrix} u \tag{5a}$$

where δ is a step input. For ease of illustration, equation (5a) may also be written in the following partitioned form:

$$\frac{d}{dt}\begin{bmatrix} [X_f] \\ [X_m] \\ \delta \end{bmatrix} = \begin{bmatrix} [F_f] & [0] & [0] \\ [0] & [M] & [B] \\ [0] & [0] & [0] \end{bmatrix} \begin{bmatrix} [X_f] \\ [X_m] \\ \delta \end{bmatrix} + \begin{bmatrix} [G_f] \\ [0] \\ [0] \end{bmatrix} u \tag{5b}$$

Relating equations (5a) and (5b) yields the following subsystem matrices:

$$F_f = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ f_{41} & f_{42} & f_{43} & f_{44} \end{bmatrix}$$ Fourth-order unaugmented follower matrix

$$\begin{bmatrix} [M] & [B] \\ [0] & [0] \end{bmatrix} = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ m_{21} & m_{22} & b_{21} \\ \hline 0 & 0 & 0 \end{array}\right]$$ Second-order model matrix (step input)

$$\begin{bmatrix} [G_f] \\ [0] \\ [0] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ g_{41} \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{Control matrix}$$

In addition, the matrices of the QPI expression (eq. (1)) for this example are given below. For convenience an error vector Y is defined as:

$$Y = HX \quad \text{Error equation}$$

The matrix H is selected so as to yield as the error the combination of the states to be minimized by the QPI. For example, if the error Y is taken to be the differences between the model and follower attitudes and rates, an H of the following form is required:

$$H = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

which yields

$$Y = \begin{bmatrix} (\phi_m - \phi_A) \\ (\dot{\phi}_m - \dot{\phi}_A) \end{bmatrix}$$

The weighting matrices of the QPI integrand, which are also selected by the user, are in this study selected as shown below.

$$Q = \begin{bmatrix} q_{11} & 0 \\ 0 & q_{22} \end{bmatrix} \qquad R = r \text{ (a scalar)}$$

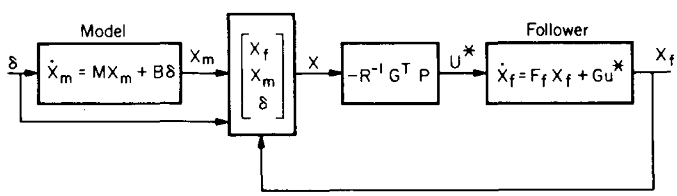

Figure 4.– Matrix representation of roll axis model-follower.

Figure 4 is a structured diagram of the system as optimized by the QPI (eqs. (1)–(5)) and involving the above matrices; U* is the optimized form of the control U. Relating this optimized form to the example in figure 3 shows that the control U may be written as

$$U = KX = -k_0\phi_A - k_1\dot{\phi}_A - k_2\ddot{\phi}_A - k_3\dddot{\phi}_A + k_{\phi_m}\phi_m + k_{\dot{\phi}_m}\dot{\phi}_m + k_\delta\delta$$

which, when optimized as given by equation (3), becomes

$$u^* = -\frac{g_{41}}{r}\begin{bmatrix} p_{41} & p_{42} & p_{43} & p_{44} & p_{45} & p_{46} & p_{47} \end{bmatrix}\begin{bmatrix} \phi_A \\ \dot{\phi}_A \\ \ddot{\phi}_A \\ \dddot{\phi}_A \\ \phi_m \\ \dot{\phi}_m \\ \delta \end{bmatrix} \tag{3a}$$

where p_{ij} is the ij element of the P matrix. Comparison of the above two expressions for u and u* reveals that in the optimized case $k_0 = (g_{41}/r)p_{41}$, $k_1 = (g_{41}/r)p_{42}$, etc. Thus solving for the appropriate p elements provides the optimal K gains, which are defined as K*.

Before proceeding with the solution of these p elements, one additional system expression will be constructed, namely, the generalized optimum follower transfer function $\phi_A(s)/\phi_m(s)$. This expression will illustrate the similarities between the p elements, the K* gains, and the follower transfer function coefficients.

This transfer function can be derived by substituting equation (3a) into line 4 of equation (5a), which yields

$$\frac{\phi_A(s)}{\phi_m(s)} = \frac{N(s)}{D(s)} = \frac{-(g_{41}{}^2/r)\left\{p_{45} + p_{46}s + p_{47}[\delta(s)/\phi_m(s)]\right\}}{\left(\begin{array}{l} s^4 + [(g_{41}{}^2/r)p_{44} - f_{44}]s^3 + [(g_{41}{}^2/r)p_{43} - f_{43}]s^2 \\ + [(g_{41}{}^2/r)p_{42} - f_{42}]s + [(g_{41}{}^2/r)p_{41} - f_{41}] \end{array}\right)} \tag{6}$$

The numerator of equation (6) can be further expanded by utilizing line 6 of equation (5a), that is,

$$\delta(s) = \frac{\phi_m(s)}{b_{21}}(s^2 - m_{22}s - m_{21})$$

which finally yields a numerator of the form

$$N(s) = \frac{-g_{41}{}^2}{r}\left[\frac{p_{47}}{b_{21}}s^2 + \left(p_{46} - \frac{m_{22}p_{47}}{b_{21}}\right)s + p_{45} - \frac{p_{47}}{b_{21}}m_{21}\right]$$

Now, by letting

$$N_1 = \frac{-g_{41}^{2}}{r} \, p_{45} - \frac{p_{47}}{b_{21}} m \tag{7a}$$

$$N_2 = \frac{-g_{41}^{2}}{r} \, p_{46} - \frac{p_{47}}{b_{21}} m_{22} \tag{7b}$$

$$N_3 = \frac{-g_{41}^{2}}{r} \, \frac{p_{47}}{b_{21}} \tag{7c}$$

$$D_1 = \frac{g_{41}^{2}}{r} p_{41} - f_{41} = g_{41} k_0 - f_{41} \tag{7d}$$

$$D_2 = \frac{g_{41}^{2}}{r} p_{42} - f_{42} = g_{41} k_1 - f_{42} \tag{7e}$$

$$D_3 = \frac{g_{41}^{2}}{r} p_{43} - f_{43} = g_{41} k_2 - f_{43} \tag{7f}$$

$$D_4 = \frac{g_{41}^{2}}{r} p_{44} - f_{44} = g_{41} k_3 - f_{44} \tag{7g}$$

the generalized fourth-order optimum follower transfer function becomes

$$\frac{\phi_A(s)}{\phi_m(s)} = \frac{N_3 s^2 + N_2 s + N_1}{s^4 + D_4 s^3 + D_3 s^2 + D_2 s + D_1} \tag{8}$$

This form will be used in the study; equations (7a) through (7g) are the key expressions showing the relationship between the p elements, the optimal gains K*, and the optimum follower transfer function coefficients.

An expression for the optimum gains will now be determined in terms of the system and weighting matrices. The steady state Riccati equation (eq. (2)) can be constructed term by term; that is,

$$0 = PGR^{-1}G^TP - PF - F^TP - H^TQH \tag{2a}$$

where

$$PGR^{-1}G^TP = \frac{g_{41}^{2}}{r} \begin{bmatrix} p_{14}^{2} & p_{14}p_{42} & p_{14}p_{43} & p_{14}p_{44} & \cdots & p_{14}p_{47} \\ p_{24}p_{41} & & & & & \\ p_{34}p_{41} & & & & & \\ p_{44}p_{41} & & & & & \\ p_{54}p_{41} & & & & & \\ p_{64}p_{41} & & & & & \\ p_{74}p_{41} & \cdot & \cdot & \cdot & & p_{74}^{2} \end{bmatrix}$$

$$
F^T P = \begin{bmatrix}
f_{41}p_{41} & f_{41}p_{42} & f_{41}p_{43} & f_{41}p_{44} & \cdot\ \cdot & f_{41}p_{47} \\
p_{11} + f_{42}p_{41} & \cdot & & & & p_{17} + f_{42}p_{47} \\
p_{21} + f_{43}p_{41} & \cdot & & & & p_{27} + f_{43}p_{47} \\
p_{31} + f_{44}p_{41} & \cdot & & & & p_{37} + f_{44}p_{47} \\
m_{21}p_{61} & \cdot & & & & m_{21}p_{67} \\
p_{51} + m_{22}p_{61} & \cdot & & & & p_{57} + m_{22}p_{67} \\
b_{21}p_{61} & \cdot & & & & b_{21}p_{67}
\end{bmatrix}
$$

Since P is symmetric, the following is true:

$$
PF = [F^T P^T]^T = [F^T P]^T
$$

$$
H^T Q H = \begin{bmatrix}
q_{11} & 0 & 0 & 0 & -q_{11} & 0 & 0 \\
0 & q_{22} & 0 & 0 & 0 & -q_{22} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-q_{11} & 0 & 0 & 0 & q_{11} & 0 & 0 \\
0 & -q_{22} & 0 & 0 & 0 & q_{22} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

In the Riccati equation, the elements of the matrices G and F are a function of the system under investigation and hence are known, and the elements of the matrices Q, R, and H are assigned by the designer. Thus only the elements of the P matrix (e.g., a 7 X 7 in the present example) must be solved for. However, it is not necessary to evaluate all 49 p elements because the P matrix is symmetric; this leaves 28 distinct p elements. Although, given a specific system, the computer conveniently provides a numerical solution for the p elements, information regarding the parametric relationships between the elements of the matrix P and the elements of matrices G, F, Q, R, and H is lost. Thus pursuing an algebraic approach, inspection of equation (3a) reveals that only the fourth row p elements need be evaluated to optimize the system. Accordingly, a generalized expression involving each of the fourth row p elements can be extracted from the Riccati equation, which is a function of G, F, Q, R, H, and also some of the other p elements. For example, the 2–2 element of equation (2a) yields an expression containing p_{42}:

$$
(-g_{41}^2/r)p_{42}^2 + 2(p_{12} + f_{42}p_{42}) + q_{22} = 0
$$

This particular expression is a function also of the element p_{12} and thus proves not to be a useful expression for solving for p_{42}. In fact, only one element, p_{41}, can be conveniently evaluated by this algebraic method. Nonetheless, the expression for p_{41} is a parametric one in terms of the system and the weighting matrices, thus satisfying one of the objectives of this report. The element p_{41} also leads to the determination of one of the more "significant" optimal gain terms, as will be subsequently shown. Solving for p_{41} is easily achieved through evaluation of the one-one element of the steady-state matrix Riccati equation, (2a); that is,

$$(-g_{41}{}^2/r)p_{41}{}^2 + 2f_{41}p_{41} + q_{11} = 0$$

Solving for p_{41} yields

$$p_{41} = \frac{rf_{41}}{g_{41}{}^2}\left[1 \pm \left(1 + \frac{q_{11}g_{41}{}^2}{rf_{41}{}^2}\right)^{1/2}\right] \quad \begin{pmatrix} \text{use - for } f_{41} < 0 \\ \text{use + for } f_{41} \geq 0 \end{pmatrix}$$

Substituting this value for p_{41} into equation (3a) yields k_0, which is the outer-loop augmenting feedback gain term that operates on ϕ_A (see fig. 3):

$$k_0 = r^{-1}g_{41}p_{41} = \frac{f_{41}}{g_{41}}\left[1 \pm \left(1 + \frac{q_{11}g_{41}{}^2}{rf_{41}{}^2}\right)^{1/2}\right]$$

Thus the optimum outer-loop augmenting feedback term for this example is a sole function of q_{11}, r, f_{41}, and g_{41}. The follower system total outer-loop gain term (i.e., the transfer function coefficient D_1) is

$$D_1 = g_{41}k_0 - f_{41} = -f_{41} + f_{41}\left[1 \pm \left(1 + \frac{q_{11}g_{41}{}^2}{rf_{41}{}^2}\right)^{1/2}\right] \tag{9}$$

In non-type-zero open-loop follower systems (i.e., a system with one or more open-loop poles at the origin) this expression can be simplified. A non-type-zero system results if $f_{41} = 0$ in figure 3

$$k_0 = (q_{11}/r)^{1/2}$$

and equation (9) becomes

$$D_1 = g_{41}(q_{11}/r)^{1/2} \tag{10}$$

This relationship indicates that for a non-type-zero fourth-order system, the optimum outer-loop gain term is a function only of g_{41}, q_{11}, and r.

N Order Example of QPI

Equations (9) and (10) were derived for a fourth-order follower system and p_{41} was determined exactly. Before the other K* gain terms are discussed, a more generalized expression

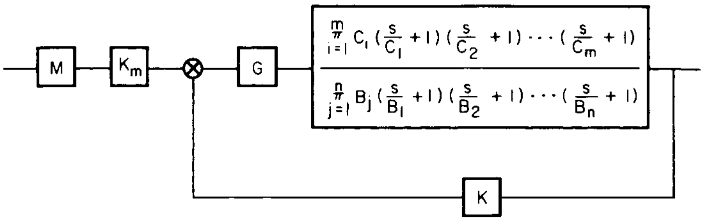

Figure 5.– N-order type-zero follower system.

of k_0 involving p_{n1} will be presented. Consider the n order type-zero system shown in figure 5. Through a development similar to that used for the fourth-order case, the optimum gain k_0 and the total outer-loop gain D_1 can be determined. With the follower system matrix F_f represented in its canonical form, the element of F which enters into the optimized k_0 expression is f_{n1}. Correspondingly, the element of G is g_{n1}, the element of Q is q_{11}, and the element of R is the scalar r. The k_0 expression can then be written as

$$k_0 = r^{-1} g_{n1} p_{n1} = \frac{f_{n1}}{g_{n1}} \left[1 \pm \left(1 + \frac{q_{11}}{r} \frac{g_{n1}^{\ 2}}{f_{n1}^{\ 2}} \right)^{1/2} \right]$$

Similarly, the total outer-loop gain term D_1 is

$$D_1 = g_{n1} k_0 - f_{n1} = -f_{n1} + f_{n1} \left[1 \pm \left(1 + \frac{q_{11}}{r} \frac{g_{n1}^{\ 2}}{f_{n1}^{\ 2}} \right)^{1/2} \right] \tag{9a}$$

where

$$f_{n1} = \prod_{i=1}^{n} B_i$$

Again, for a system with $f_{n1} = 0$ (non-type-zero), equation (9a) reduces to

$$D_1 = g_{n1} (q_{11}/r)^{1/2}$$

Table 1 summarizes the analytical expressions for the n order case.

TABLE 1.– ANALYTICAL SUMMARY – N ORDER OPTIMAL SYSTEMS

	Zero-Type Systems*	Non-Type-Zero Systems
Outer-loop augmenting gain k_0	$\frac{f_{n1}}{g_{n1}} \left[1 \pm \left(1 + \frac{q_{11} g_{n1}^{\ 2}}{r f_{n1}^{\ 2}} \right)^{1/2} \right]$	$\left(\frac{q_{11}}{r} \right)^{1/2}$
Total outer-loop gain (TOLG = D_1)	$-f_{n1} + f_{n1} \left[1 \pm \left(1 + \frac{q_{11} g_{n1}^{\ 2}}{r f_{n1}^{\ 2}} \right)^{1/2} \right]$	$g_{n1} \left(\frac{q_{11}}{r} \right)^{1/2}$

* Use – for f_{41} less than 0.
Use + for f_{41} greater than or equal to zero.

Since the X–14 airplane equations are non-type-zero, fourth-order non-type-zero systems are considered in the remainder of this report. Hence, the outer-loop augmenting gains will be $(q_{11}/r)^{1/2}$, and the total outer-loop gains will be $g_{41}(q_{11}/r)^{1/2}$.

As indicated previously, unlike p_{41} the outer p elements, and thus the other model feedforward and follower feedback terms, are not as easily expressed analytically through the use of the Riccati equation. Thus a graphical method was developed that utilizes the optimum outer-loop gain, which is the transfer function coefficient D_1, and relates all the other optimum transfer function coefficients to it.

The purpose of this graphical investigation is to show the relationship between D_1 (hence, f_{n1}, g_{n1}, q_{11}, and r) and the remaining system optimum loop gains. Only the fourth-order follower system with constant numerator will be treated here although this development does not preclude its use for other order systems. Initially for illustrative purposes a simple system whose transfer function is $1/s^4$ will be considered for both the QPI attitude weighting case and then for the attitude and rate weighting case. Through optimization of this system for several Q and R matrices, graphical asymptotes can be constructed which, under certain conditions, become the asymptotes for other fourth-order systems. Further, for the attitude weighting only case, these asymptotes represent the Butterworth filter form coefficients. After the parametric asymptotes are established for the $1/s^4$ case, the graphical technique will be used to optimize the X–14B system.

Attitude Weighting Only

To show the relationship between g_{41}, q_{11}, r, and the inner-loop terms, first consider the preceding generalized fourth-order follower system in figure 3 for the specific case $f_{41}=f_{42}=f_{43}=f_{44}=0$ (i.e., transfer function of $1/s^4$). The example with this follower does not have much practical significance, but is used to show that the following graphical technique obtains the Butterworth functional relationships that are known to exist for this case when the QPI is optimized (refs. 5 and 17).

The optimal gains for the $1/s^4$ follower can be determined parametrically without the aid of a computer by utilizing a direct solution detailed in reference 15. This method centers on the algebraic solution of the expression

$$\left| Is - (F - GK) \right| \left| -Is - (F - GK)^T \right| = \begin{vmatrix} Is - F & GR^{-1}G^T \\ -H^TQH & -Is - F^T \end{vmatrix}$$

After the determinants in the above equation are expanded and coefficients of like powers of s are equated from the left and right side of the equation, the following optimal gains result:

$$k_0 = (q_{11}/r)^{1/2}$$

$$k_1 = (2.6/g_{41}{}^{1/4})k_0{}^{3/4}$$

$$k_2 = (3.4/g_{41}^{1/2})k_0^{1/2}$$

$$k_3 = (2.6/g_{41}^{3/4})k_0^{1/4}$$

For the $1/s^4$ example, the optimal transfer function of the form of equation (8) simplifies to

$$\frac{\phi_A(s)}{\phi_m(s)} = \frac{N_3 s^2 + N_2 s + N_1}{s^4 + g_{41}k_3 s^3 + g_{41}k_2 s^2 + g_{41}k_1 s + g_{41}k_0}$$

Hence the above optimal gain expressions for k_0, k_1, k_2, and k_3 yield the following transfer coefficients:

$$D_1 = g_{41}k_0$$

$$D_2 = g_{41}k_1 = 2.6\ D_1^{3/4}$$

$$D_3 = g_{41}k_2 = 3.4\ D_1^{1/2}$$

$$D_4 = g_{41}k_3 = 2.6\ D_1^{1/4}$$

Note that for this case the exponents and numerical coefficients are Butterworth.

Although the analytical technique yields the parametric relationships for the $1/s^4$ follower, it may be tedious to apply in other cases. For those cases, a simple-to-apply graphical technique has been developed. The technique is based on the observation that the parametric relationships as calculated analytically for the $1/s^4$ plant are of the general form

$$D_n = b\ D_1^m$$

This equation, when represented in log-log coordinates, is represented by a straight line, where constants b and m represent the intercept and slope of the line.

Now pursuing the graphical approach, the $1/s^4$ follower system was optimized through the solution of the Riccati equation for several values of q_{11} and r. The p elements for the special case of $q_{11} = 10$, $q_{22} = 0$, $r = 0.01$, and $g_{41} = 1$ (i.e., $q_{11}/r = 1000$) resulted in the following optimum control (see eq. (3a)):

$$u^* = -100[0.316 \quad 0.348 \quad 0.192 \quad 0.062 \quad -0.069 \quad -0.042 \quad -0.092]X$$

where, from this computer solution, the total outer-loop gain term, D_1, is

$$D_1 = g_{41}k_0 = 31.6$$

This can also be obtained analytically by utilizing equation (6); that is,

$$D_1 = \frac{g_{41}^{\ 2}}{r}p_{41} = g_{41}\left(\frac{q_{11}}{r}\right)^{1/2} = 31.6$$

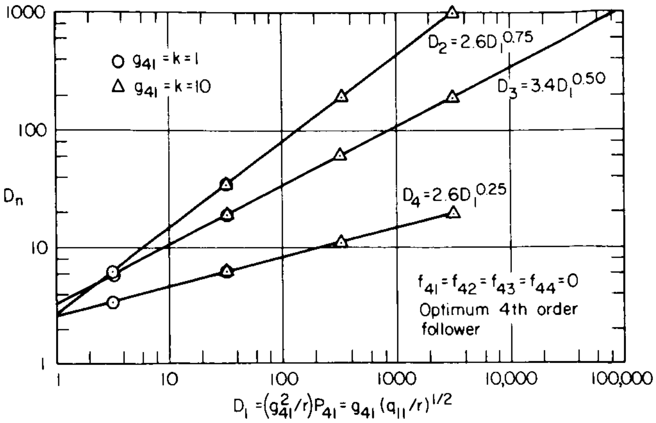

Figure 6.– Optimum denominator terms for $1/s^4$ plant (attitude weighting only).

A plot of this computer solution plus solutions for other D_1 terms yielded the log-log plot (fig. 6) of the individual denominator terms D_4, D_3 and D_2 plotted versus D_1. The values of $g_{41}(q_{11}/r)^{1/2}$ plotted were 3.162, 31.62, 316.2, and 3162 with $g_{41} = 1$ and $g_{41} = 10$. This log-log plot might well have been p_{44}, p_{43}, and p_{42} plotted versus p_{41} and the same straight-line functional relationship would still exist. However, the quantities used in figure 6 are preferred for substitution into the optimum transfer function of equation (8); that is, the optimum transfer function coefficients for any q_{11}/r can be determined from this representation.

Figure 6 indicates that the relationships between the optimum total outer-loop gain and all of the optimum inner-loop gains can be exactly characterized by the equation shown earlier,

$$D_n = bD_1{}^m$$

For example,

$$D_2 = 2.6D_1{}^{0.75}$$

which, as anticipated, is the same parametric expression previously generated by the direct solution.

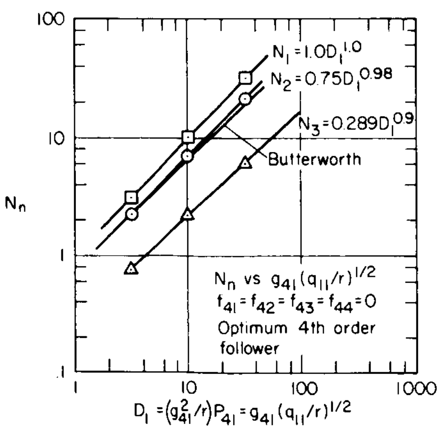

Figure 7.– Optimum numerator terms for $1/s^4$ plant (attitude weighting only).

In figure 7 the individual numerator terms in equation (8) (N_1, N_2, and N_3) are plotted against the same denominator term D_1. This log-log plot indicates the same straight-line functional relationship seen in figure 6.

Six functional relationships for this system evolve from figures 6 and 7:

$$D_2 = 2.6D_1{}^{0.75}$$
$$D_3 = 3.4D_1{}^{0.50}$$
$$D_4 = 2.6D_1{}^{0.25}$$
$$N_1 = D_1$$
$$N_2 = 0.75D_1{}^{0.98}$$
$$N_3 = 0.289D_1{}^{0.90}$$

These significant relationships allow the optimum system transfer function in equation (8) to be written as a function of only D_1. Thus

$$\frac{\phi_A(s)}{\phi_m(s)} = \frac{0.289D_1^{0.9}s + 0.75D_1^{0.98}s + D_1}{s^4 + 2.6D_1^{0.25}s^3 + 3.4D_1^{0.5}s^2 + 2.6D_1^{0.75}s + D} \tag{11}$$

where it will be recalled that

$$D_1 = \frac{g_{41}^2}{r} p_{41} = g_{41}\left(\frac{q_{11}}{r}\right)^{1/2}$$

It is sometimes convenient to express equation (11) as

$$\frac{\phi_A(s)}{\phi_m(s)} = \frac{0.289D_1^{0.9}s^2 + 2.6D_1^{0.08}s + 3.46D_1^{0.1}}{s^4 + 2.6D_1^{0.25}s^3 + 3.4D_1^{0.5}s^2 + 2.6D_1^{0.75}s + D_1} \tag{12}$$

Four observations can be made concerning this example.

1. The denominator is exactly of Butterworth form; that is,

$$s^4 + 2.6\omega s^3 + 3.4\omega^2 s^2 + 2.6\omega^3 s + \omega^4 \tag{12a}$$

This, in general, was anticipated because it has been shown that for systems with one input and a scalar Q, the QPI produces an optimum system having a Butterworth response (refs. 5 and 17). Equation (11) indicates that optimization of this particular system produces a Butterworth form for all values of q_{11} and r. It will be shown later that open-loop system dynamics greatly influence the magnitude of q_{11}/r required to produce a Butterworth response from the closed-loop system (see fig. 13).

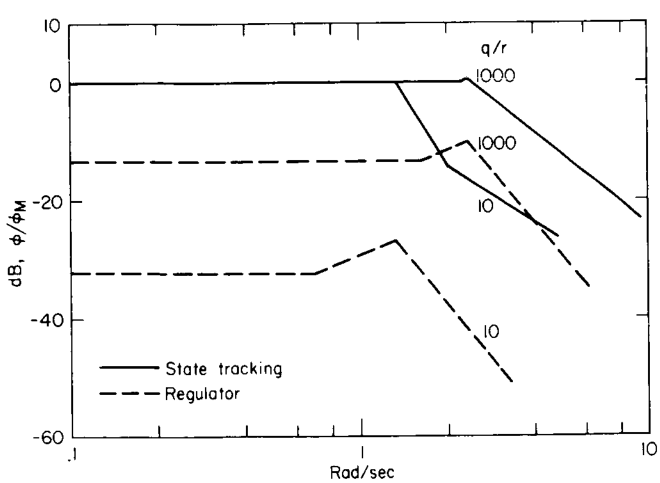

Figure 8.– Bode plot comparing optimum state tracking and regulator cases for q/r = 10 and q/r = 1000.

2. The transfer function resulting from the state tracking optimum systems always indicated unity dc gains for ϕ_A/ϕ_m (e.g., see eq. (11)); however, transfer functions resulting from regulatory optimums did not produce unity dc gains (see ref. 5 and fig. 8). The feedback gain terms produced by the QPI for the regulatory optimums were the same as the corresponding terms for the state tracking, but the state tracking cases had one more feedforward gain term than the regulatory cases. This extra gain term is on the input and the input can be reformulated into other feedforward states plus acceleration. It was the

addition of the extra term on the attitude feedforward from the input that resulted in optimums with unity dc gains. Applications of the QPI requiring unity dc gain should not be optimized in the regulator form; however, the state tracking form can be used to obtain unity dc gain.

3. The numerator terms as expressed in the form given by equation (8) also yielded straight-line log-log plots.

4. The Butterworth numerator coefficient for N_2 plotted on figure 7 is observed to be a good approximation for the computer derived values for N_2 over this range of D_1.

The calculations of this example will now be repeated but with a weighting factor on the rate error.

Attitude and Rate Weighting Only

When the system of figure 3 with $f_{41} = f_{42} = f_{43} = f_{44} = 0$ was optimized for both attitude and rate weighting for several combinations of r, g_{41}, and with $q_{11} = q_{22}$ (i.e., equal weights on both attitudes and rates), straight-line log-log plots for D_4, D_3, D_2, versus D_1 again resulted. Because of the effect of q_{22} on the system, the optimum coefficients for this case were no longer Butterworth. Nonetheless, as shown in figures 9 and 10, the coefficients may be expressed by the same form, that is,

$$\left.\begin{aligned} D_4 &= 2.68D_1^{0.28} & N_1 &= D_1 \\ D_3 &= 3.55D_1^{0.575} & N_2 &= 0.75D_1^{1.1} \\ D_2 &= 2.74D_1^{0.84} & N_3 &= 0.31D_1^{1.0} \end{aligned}\right\} \qquad (13)$$

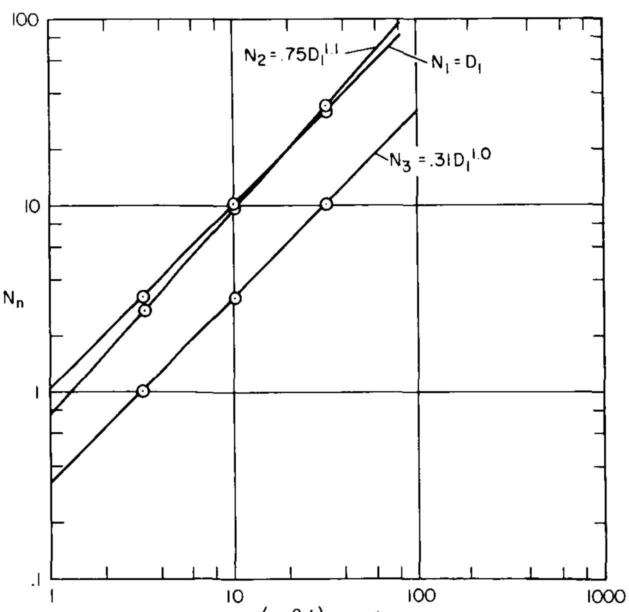

Figure 9.– Optimum numerator terms for $1/s^4$ plant (attitude and rate weighting).

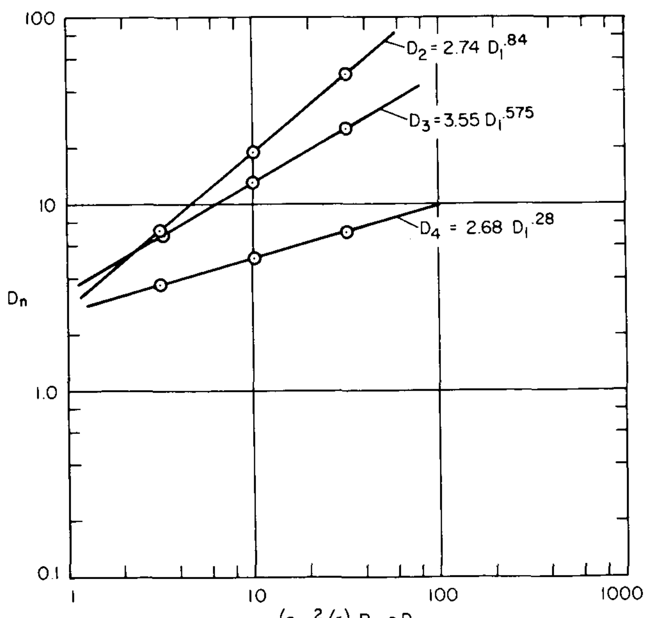

Figure 10.– Optimum denominator terms for $1/s^4$ plant (attitude and rate weighting).

Equations (13) provide the optimum follower transfer function for the attitude and rate weighting as follows:

$$\frac{\phi_A(s)}{\phi_m(s)} = \frac{0.31D_1s^2 + 0.75D_1^{1.1}s + D_1}{s^4 + 2.68D_1^{0.28}s^3 + 3.55D_1^{0.575}s^2 + 2.74D_1^{0.84}s + D_1}$$

The important feature here is that all the terms of both preceding examples are characterized by a similar form of parametric relationship. All feedback and feedforward gains have a functional relationship to D_1 for attitude and rate weighting as well as attitude weighting only.

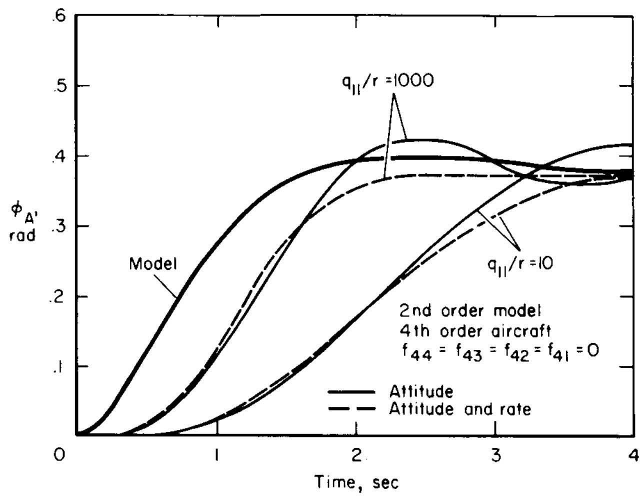

Figure 11.– Attitude time response for $1/s^4$ plant comparing attitude weighting to attitude and rate weighting.

It should be noticed that the total outer-loop gain term, D_1, is not influenced by the addition of q_{22}. However, as shown by the equations for D_2, D_3, and D_4 (eqs. (13)), q_{22} does influence the inner-loop optimum coefficients and thus the dynamics of the follower response. This characteristic can perhaps be best shown by figure 11, where the optimum system attitude time responses are shown for the follower transfer function of $1/s^4$ that has been optimized by the QPI. In figure 11, time histories of attitude and rate weighted cases (i.e., $q_{11} = q_{22}$) are visibly more damped than are the attitude weighted cases (i.e., $q_{22} = 0$). However, if q_{11} and q_{22} are altered a faster responding system can be produced. Figure 11 indicates some of the flexibility within the QPI for producing various system responses.

The calculations for the single-axis attitude weighting only example will now be repeated for the X–14B aircraft system.

X–14B System

To extend the dependency of the feedforward and inner-loop feedback terms on D_1, consider a more physically realistic system in which not all the follower feedback terms f_{41}, f_{42}, f_{43}, and f_{44} are zero. Such a system is shown in figure 3. The X–14B example will be of the form of figure 3, but with $f_{41} = 0$.

The plant representation of the X–14B, as described in Guidelines for Investigation and appendix B, consisted of an actuator and single-axis aircraft equation:

$$\frac{\phi_A(s)}{u(s)} = \frac{L_\delta/I_x}{s(s + L_p/I_x)} \times \frac{\omega_a^2}{s^2 + 2\zeta_a\omega_a s + \omega_a^2}$$

$$= \frac{4.585}{s(s + 0.45)} \times \frac{62.8^2}{s^2 + 2(0.5)(62.8)s + 62.8^2}$$

$$= \frac{18,100}{s(s^3 + 63.3s^2 + 3978s + 1775)} \tag{14}$$

where $(L_\delta/I_X)\omega_a^2 = g_{41}$ is defined in appendix B. With the addition of system dynamics, the direct solution shown in the first example becomes considerably more laborious; thus only the graphical solution is presented here. The system transfer function will now be optimized for various attitude weightings. Since the system transfer function is fourth order, the optimum transfer function can be expressed in the form given by equation (8):

$$\frac{\phi_A(s)}{\phi_m(s)} = \frac{N_3 s^2 + N_2 s + N_1}{s^4 + D_4 s^3 + D_3 s^2 + D_2 s + D_1}$$

where, from table 1, the total outer-loop gain is equivalent to D_1 and

$$D_1 = g_{41}(q_{11}/r)^{1/2}$$
$$D_1 = 4.585(62.8)^2(q_{11}/r)^{1/2}$$

Figure 12 indicates the manner in which the asymptotes for the D_2, D_3, and D_4 denominator coefficients can be drawn. Equations (7e), (7f), and (7g) indicate that the exact values for D_2, D_3, and D_4 are:

$$D_2 = g_{41}k_1 - f_{42}$$

$$D_3 = g_{41}k_2 - f_{43}$$

$$D_4 = g_{41}k_3 - f_{44}$$

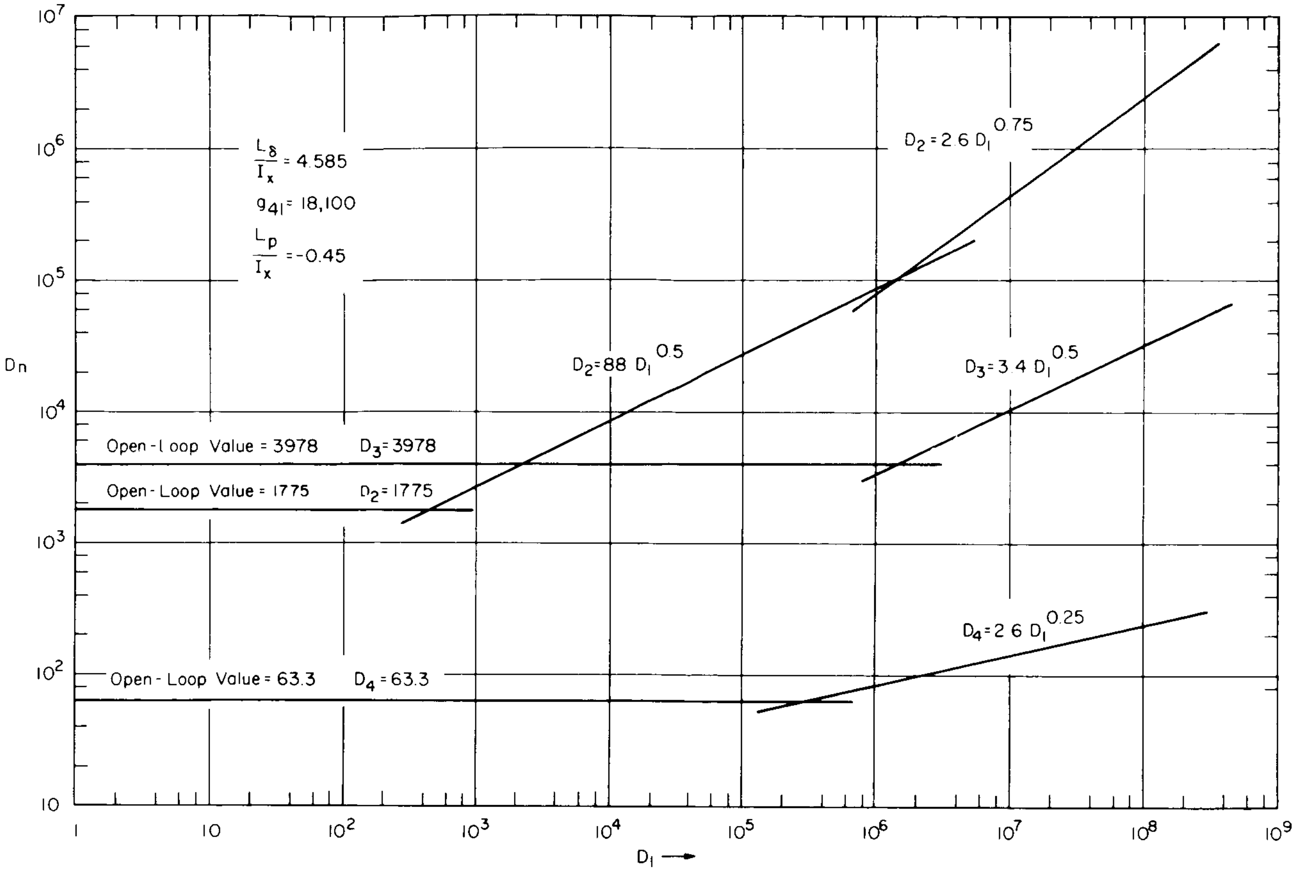

Figure 12.– Denominator asymptotes for X–14B roll axis (attitude weighting).

The asymptotes for the denominator coefficients D_4, D_3, and D_2 will, for every small values of augmenting gains, be the open-loop system values of f_{44}, f_{43}, and f_{42}. Hence $D_4 = 63.3$, $D_3 = 3978$, and $D_2 = 1775$ are the asymptotes for this example. Next the asymptotes for D_4, D_3, and D_2 will be determined for the region where the augmenting gains are very large relative to the open-loop system values. Because the example uses attitude weighting only, as augmenting gains increase without limit, the optimal transfer function coefficients, D_2, D_3, and D_4, approach the Butterworth coefficient asymptotes of $2.6D_1^{0.75}$, $3.4D_1^{0.5}$, and $2.6D_1^{0.25}$, respectively (see ref. 17). Between the small and the large augmenting gain regions, additional midrange asymptotes can sometimes be drawn. Midrange asymptotes can be drawn in our example because the 10–cps actuator dynamics are much greater than the other system dynamics and could be eliminated in the system representation for small augmenting gain values. Hence, $\phi_A(s)/u(s) = (L_\delta/I_x)/s[s + (L_p/I_x)]$ would be the open-loop transfer function being optimized if the 10–cps actuator transfer function were considered to be unity. The asymptote for this optimized second-order open-loop transfer function for attitude weighting is a second-order Butterworth transfer function; that is,

$$\frac{\phi_A(s)}{u(s)} = \frac{D_1'}{s^2 + 1.4(D_1')^{0.5}s + D_1'} \tag{15}$$

where, from table 1,

$$D_1' = g_{41}'(q_{11}/r)^{1/2}$$

To plot these second-order asymptotes for D_2 on the fourth-order graph of figure 12, equation (15) must be rewritten in terms of D_1. Replacing the actuator of equation (14) by unity results in a numerator g_{41}' of

$$g_{41}' = L_\delta/I_X$$

hence

$$g_{41}' = g_{41}/\omega_a^2$$

and therefore the second-order D_1' equals the fourth-order D_1 divided by ω_a^2, that is,

$$D_1' = D_1/\omega_a^2$$

Substituting this expression into equation (15) yields a transfer function in terms of D_1,

$$\frac{\phi_A(s)}{u(s)} = \frac{D_1/\omega_a^2}{s^2 + 1.4(D_1/\omega_a^2)^{0.5}s + D_1/\omega_a^2}$$

$$\frac{\phi_A(s)}{u(s)} = \frac{D_1}{\omega_a^2 s^2 + 1.4\omega_a D_1^{0.5}s + D_1}$$

$$= \frac{D_1}{3945s^2 + 88D_1^{0.5}s + D_1}$$

Thus, the $88D_1^{0.5}$ term becomes the midrange asymptote for D_2.

The preceding discussion indicates how asymptotes for denominator coefficients in the optimum transfer function can be drawn for all values of D_1. The asymptotes in figure 12 should be a good approximation of the optimum coefficient values for attitude weighting only. Figure 13 is a plot of the optimum solutions as well as the asymptotes previously shown in figure 12. It can be seen that over a considerable range of the variable D_1, the asymptotes provide a good approximation of the true optimums.

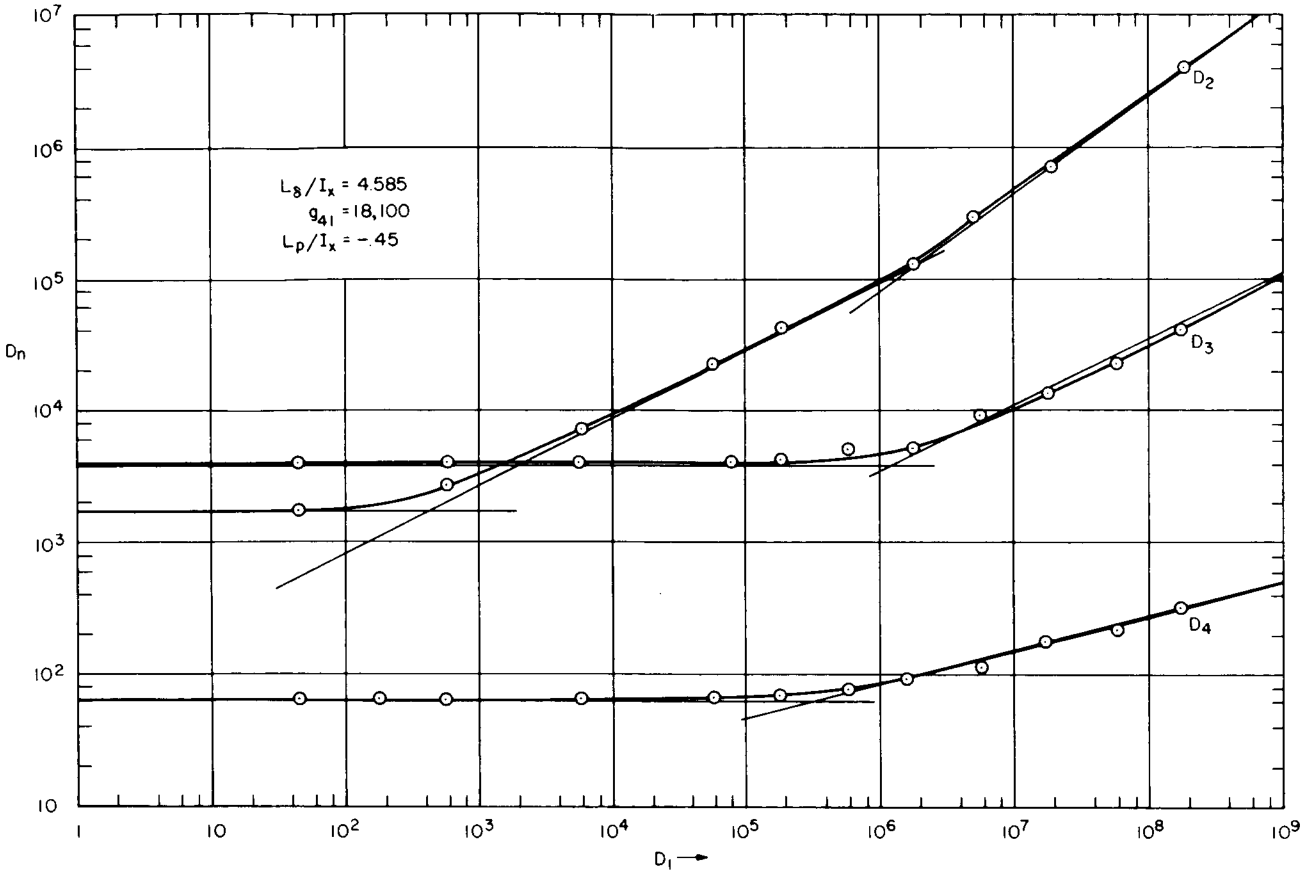

Figure 13.– Denominator terms for X–14B roll axis (attitude weighting).

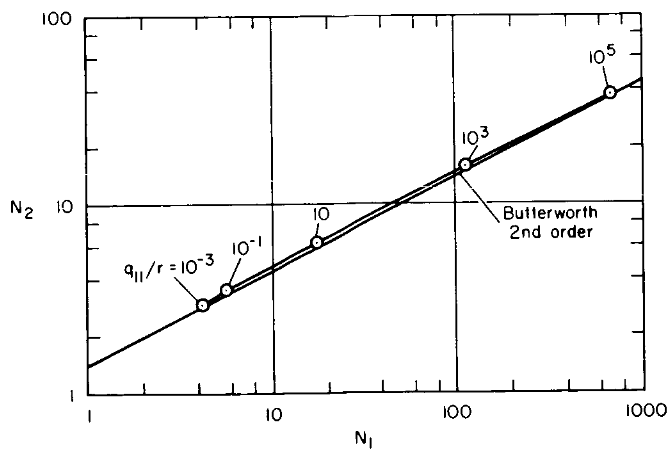

Figure 14.– Numerator terms for X–14B roll axis (attitude weighting).

An asymptote for the optimum numerator coefficient N_2 can be found in much the same way as the denominator coefficients were determined. Figure 14 is a plot of the numerator coefficient asymptotes and the actual optimum values.

CONCLUSIONS

The quadratic performance index was found to be the most versatile criterion for the design of a prefilter model reference system for a VTOL aircraft. Calculation of the optimum system gains for this criterion can be determined through readily available digital computer programs. The system, which was optimized in this study, consists of a second-order prefilter model and a simplified fourth order (single input, single control, constant numerator, and non-type-zero) roll axis representation of the X–14B aircraft.

Expressions for the optimum total outer-loop gain were analytically determined for both type zero and non-type-zero systems. Analysis of the canonical form of the state equation and the Riccati equation for single-input, single-control systems indicates that the total-outer-loop gain can always be analytically determined. For type-zero systems the total-outer-loop gain includes the outer-loop weighting element of the Q matrix (the 1–1 element in the report examples), the control weighting element of the R matrix, the total system gain element, and the outer-loop gain term of the unaugmented follower. As the outer-loop gain term of the unaugmented follower approaches zero the optimum total-outer-loop gain approaches that of the non-type-zero system, as was expected. For the non-type-zero system the total-outer-loop gain is the product of the total system gain and the square root of the quotient of the 1–1 element of the Q matrix divided by the scalar element of R (only single-input, single-control systems were studied).

The only terms that affect the optimum outer-loop feedback gains are the attitude weighting term, the control weighting term, the total system gain term, and, in the case of the type-zero system, the outer-loop gain term in the unaugmented follower. Hence the addition of other weighting terms (such as rate weighting) does not affect the outer-loop feedback gain. Weighting terms other than on attitude will, however, influence the inner-loop dynamics of the follower.

A graphical method for approximating the optimal gains was developed. By this method asymptotes were generated for the inner-loop feedback gains in terms of the total outer-loop gain over a wide range of weighting. The asymptotes approximate the loop gain values for the inner feedback and the feedforward terms. For the X–14B example, with attitude weighting only, the asymptotes plotted on log-log coordinates are piecewise linear. For small weighting of attitude the asymptotes represent the characteristics of the unaugmented X–14B, and for large weighting of attitude the asymptotes are of the fourth-order Butterworth form. For this particular example, an additional set of mid-range asymptotes result which are of the second-order Butterworth form. With the addition of equal rate weighting in the performance index, straight-line asymptotes can again be constructed – however, not of Butterworth form. The asymptotes for this and other combinations of weighting are constructed with the aid of a digital computer.

Applications of the quadratic performance index requiring unity dc gain should not be optimized in the regulator form; however, the state tracking form can be used to obtain unity dc gain.

Ames Research Center
National Aeronautics and Space Administration
Moffett Field, Calif. 94035, Aug. 18, 1970

APPENDIX A

HISTORICAL REVIEW

This appendix briefly reviews the background of three areas of research investigated in this study: (1) characteristics of different performance criteria, (2) desirable handling qualities models, and (3) model reference systems.

Various performance criteria were studied in references 8 to 13. In principle these criteria can be used to optimize any order or type system; however, their effectiveness depends on whether or not the chosen error criterion is the one to be minimized. What is considered optimum in one application might not be considered optimum in another. This section of the appendix will discuss some of the considerations for using performance criteria for model reference systems rather than for the conventional response feedback system.

A prefilter model follower system may, in many ways, be considered an extension of a conventional response feedback system. To illustrate this extension, consider a single-input, single-output response feedback system. Such a configuration can be considered a model follower with a unity model, in which the system (or follower) is attempting to follow the model (the input). In other words, the response feedback configuration contains a model with infinite bandwidth and thus a follower to model bandwidth ratio of zero. A model with a less than infinite bandwidth would then introduce a nonunity transfer function as a model; hence the scheme would take on a more conventional model following form. A decrease in the model bandwidth would increase the follower to model bandwidth ratio and hence improve the ability of the follower to match the model's response.

Because of this rough similarity, performance criteria often cited in response feedback studies were the first to be considered for application to model reference systems. The major performance criteria discussed in these studies are:

$I_1 = \int_0^T \epsilon \, dt$	Integral of error
$I_2 = \int_0^T \lvert \epsilon \rvert dt$	Integral absolute error
$I_3 = \int_0^T \epsilon^2 \, dt$	Integral error squared
$I_4 = \int_0^T t \lvert \epsilon \rvert \, dt$	Integral time absolute error
$I_5 = \int_0^T t_2^2 \epsilon^2 \, dt$	Integral time squared error squared
$I_6 = \int_0^T F[\epsilon, t] \, dt$	Integral weighted error

$$I_7 = \int_0^T [\epsilon^2 + t_1\dot{\epsilon}^2 + \ldots + t^{2n}(d^n\epsilon/dt^n)^2]dt$$ Generalized performance index

$$I_8 = \int_0^T [\bar{\epsilon}^T Q \bar{\epsilon} + \bar{u}^T R \bar{u}]\, dt$$ Quadratic performance index

Each criterion had some advantages and some deficiencies, and the proper criterion for a given system optimization should be chosen in light of its characteristics. Criteria I_1 through I_6 are usually applied to single error terms. Criteria I_2, I_4, and some forms of I_6 and I_7 are difficult to evaluate analytically. Criterion I_8 was chosen because it includes the cost of control, produces a closed form solution for a multiparameter optimization integral, and can be expeditiously optimized through a digital computer program. The indicial response characteristics for response feedback systems for each of these criteria is discussed in the following paragraphs. For criteria I_2, I_3, I_4, and I_8 comments regarding optimums produced for model reference systems are included. The prefilter model is that described in the section on desirable handling qualities.

Criterion I_1 is quite adversely affected when used to optimize systems that overshoot because the overshoots decrease rather than increase the integral. This criterion cannot be used satisfactorily in producing an optimum second-order system. (The minimum I_1 is for zeta = 0.0.)

Criterion I_2 has moderate selectivity in optimizing systems up to sixth order, and is intuitively appealing since it represents the area of error between the model and the follower time histories. Optimization through the use of this criterion leads to a minimization of this area. While absolute value functions are very difficult to use analytically, they are easily mechanized on analog or digital computers. For the second-order system, the optimum ζ is 0.7. When used with model reference systems, I_2 produced very satisfactory, nonoscillatory follower responses. Followers of second, fourth, and sixth order were studied with repetitive operation analog computers.

Criterion I_3 is the most widely used of all the criteria primarily because of its mathematical convenience; however, systems thus optimized tend to be more underdamped than do systems optimized by other criteria. I_3 can be used with statistical inputs and Parseval's integral technique has been used in evaluating this integral exactly. Variations of this integral, such as I_5, can be evaluated by Parseval's technique; however, system orders greater than five are very cumbersome to evaluate. When used with model reference systems I_3 produced unsatisfactory follower responses. The optimum I_3 gains produced follower responses that exhibited the same oscillatory nature as were exhibited for the response feedback systems. Hence this criterion was rejected for application on the X–14B aircraft.

Criterion I_4 has been lauded by Graham and Lathrop (ref. 8) as a criterion with better selectivity than I_2 or I_3, and one that picks "good" systems; however, it is difficult to evaluate analytically. For a response feedback system I_4 has been found most suitable when optimizing to a step input (see ref. 8). The time weighting in the integrand tends to reduce the cost of the large initial error between the step input and the follower output, which is inherent in the response feedback system. However, for a model following system no such large initial error should exist between the model and the follower. In fact, for a step input into the model, the follower should follow the model "closely" at all times. Thus, for such a system, the time weight feature of I_4 serves no useful function and hence its use was not considered further.

Criterion I_5 yields step responses that are less oscillatory than I_3, and compares favorably with the more desirable system optimums provided by I_4. While I_5 is more cumbersome than I_3, it can still be evaluated analytically.

Criterion I_6 is a comprehensive representation for any of the above criteria and has extra provisions for special error weightings to be chosen according to the particular design objectives. For example, special functions might penalize overshoots more than undershoots.

Criterion I_7, a generalized performance index proposed by Aizerman (ref. 18), contains the system error, its first n time derivatives, and the constants τ_i, where $i = 1, 2, \ldots n$, determined from the differential equation of the desired system response. This criterion is included because it leads into I_8 and gives some insight into the choice of the matrix elements of I_8.

Criterion I_8 is the quadratic performance index explained in appendix B. It has several significant advantages, one being that it includes the cost of control in the integral. The ability to weight the various states of the solution enables the designer to choose the proper weighting matrices to provide the particular desired responses. That is, if the obtained response is too oscillatory, then through the weighting matrix element changes the desired, less oscillatory, response can be obtained. The quantity Q/R should be chosen according to the desired bandwidth ratio of the follower to the model. The disadvantages of the QPI include the following: the index must be expressed only in terms of states and the control vector, all states must be observable, only linear system formulations are acceptable. Even with these disadvantages, it is felt that I_8 is the best performance criterion.

Various desirable models were studied in references 1, 3, 6, 7, 19, and 20. The model chosen for this study was the roll axis for a VTOL aircraft which had been optimized for handling qualities in hover. The model, expressed in terms of the desired aircraft roll attitude, represents the optimum attitude control system determined on a large moving base simulator for a variety of tasks (see ref. 7). The criterion optimized in reference 7 to obtain the desired model was pilot opinion; hence an implicit and unknown weighting was assigned by the pilot to roll attitude, rate, acceleration, lateral translation, stick forcing functions required, controlled element dynamics, and the manipulator. Other variables that affected the rating were the visual scene, g-levels, training, fatigue, motivation, illumination, vibration and temperatures. In reference 7, the optimum attitude system in roll was found to require a damping of −2.8/sec and a natural frequency of 2 rad/sec. As a matter of interest, this system also required a control power in excess of 1.2 rad/sec^2 and a stick sensitivity of 0.5 rad/sec^2/in. This system yielded an average pilot rating of 2 on the Cooper Pilot Rating Scale (see ref. 21). Simulated conditions were for calm air; hence to extend results to flight, additional control power must be included for trim and upset disturbances. The roll manipulator had a maximum control deflection of ±5 inches with a force gradient of 1.8 lb/in. and a breakout force of 1 lb.

For comparison, the hover model used for the Bell X–22 aircraft was $\omega = 3$ rad/sec, and $\zeta = 1.0$ (ref. 6). The pitch axis in the X–15 uses a model of $\omega = 2$ rad/sec and $\zeta = 0.7$ for the rate command (ref. 19).

Model reference systems are discussed in references 1, 3, 4, 5, 6, 22, 23, and 24. Several types of model reference systems have been discussed in the literature, primarily the following: (1) prefilter model following systems (sometimes called the model in the system or the high gain system), (2) implicit model or model in the performance index systems, (3) inverse model systems, and (4) parameter adjustment systems. All of these approaches with the exception of (2) are considered to be adaptive, with the gains either slewed or fixed. The intended application of a model reference system is the chief determinant of the best system configuration. The following paragraphs review some characteristics of four types of model reference systems.

The configuration under consideration in this study is the prefilter model following system with fixed gains. The follower used in these examples was a VTOL hovering aircraft. It was also assumed that an onboard airborne computer on which to program the model would be available.

The prefilter model following system requires a physical or programmed model to which the controlled system or follower alines itself. If the system to be optimized is to operate over a wide range of conditions, then it may be desirable for the follower to be altered. An example of a prefilter model following system using this approach is the NASA–FRC Lockheed Jetstar (refs. 1, 3, 4, 22) in which dynamic pressure can be used to vary the appropriate gain terms on the errors between the model and corresponding aircraft parameters. Varying or slewing of gain need not be restricted to the follower but may include a variable model as well. A possible application of this most general case is a VTOL aircraft going from hover to cruise, in which case the desirable model to be simulated is continuously changing.

One primary limitation of the prefilter model following system is the operational problem associated with obtaining sufficient closed loop follower bandwidth. Factors affecting bandwidth include sensor and system noise, structural vibration feedback, nonlinearities, and sensor and actuator dynamics. These effects, however, will also degrade the performance of the other approaches.

For "good" following, the follower bandwidth should be roughly twice the model bandwidth (refs. 1, 22). However, the high-frequency gust response effects on the pilot for this type of system are not clear. Although these effects have not been a problem with this type system in the X–15 (ref. 19), it should be noted that the X–15 disturbance content is not as high as that for a VTOL. Another limitation of the model following system is that the complexity of the system to be modeled generally requires the use of an airborne computer.

An advantage of the prefilter model following system is its ability to achieve both static and dynamic matching of the follower to the model. Other desirable features of this system are its ability to null the effects of a follower with either poorly defined or slightly variable parameters and its inherent ability to isolate or decouple the axis of control.

The model in the performance index is a nonadaptive technique that provides a way of matching desired dynamics. In implementing this technique, the model is not available for reference when the system is operating, because the system has already been optimized and the model removed. For example, if changes in the plant occur after the system has been optimized, then the uncorrected effect is felt on the plant outputs. The advantages of this technique are simplicity and low cost of implementation. For instance, a computer on which to program the model is not required.

The inverse model in the feedback approach uses a high gain and some compensation in the forward loop so that the closed-loop transfer function approaches the ratio of the forward transfer function over the open-loop transfer function. In the limit, the closed-loop transfer function becomes the reciprocal of the feedback transfer function (which is the inverse of the model). Thus the closed-loop system becomes the desired model. The model in this approach usually has more poles than zeros; hence the inverse requires at least one differentiation that introduces noise into the system. An operational flight test of a high gain saturating version of this approach at Ames on an F–102 was successful (ref. 25). The test was conducted for Mach numbers from 0.36 to 1.15 and altitudes from 10,000 to 40,000 feet. The system limited longitudinal natural frequencies to one-half the basic aircraft frequencies, and the damping ratios were improved to a range from 0.56 to 0.69. However, objectionable features were sensitivity to gust disturbances and high θ to δ elevator gains at low speeds.

In the model reference adaptive parameter adjustment technique, an adjustment mechanism is used to alter compensation elements so that the form of the overall system is the same as that of the model. The parameter adjustment techniques can be grouped into three general categories in which adjustment of the compensation systems is based on the gradient of the error functions, Lyapunov's direct method, and partial derivatives of functions. In all these approaches, a nonzero input is necessary for adjustment. With no inputs, errors between the actual and desired values can result. In both the first and third categories, the error function has a large effect on the proper operation of the adaptive portion of the system.

APPENDIX B

MATHEMATICAL FORMULATION

The techniques of state space modeling and of minimizing the quadratic performance index are detailed within this section.

State Space Formulation

Figure 15 illustrates a composite block diagram of the type system considered in this report. The follower is expanded into three cascaded second-order transfer functions (i.e., actuator, bending mode, and linearized roll hover equation). In this diagram, each gain block (e.g., K_m, K_C, K_B, K_A) represents an array of scalar gains that operates on all of the states of the model and follower. When devoid of the bending mode, this system reduces to the fourth-order configuration considered in the test. A further simplification such as approximating the actuator by a unity gain would reduce the follower to second order and thus allow for the application of perfect model following techniques (refs. 15 and 24).

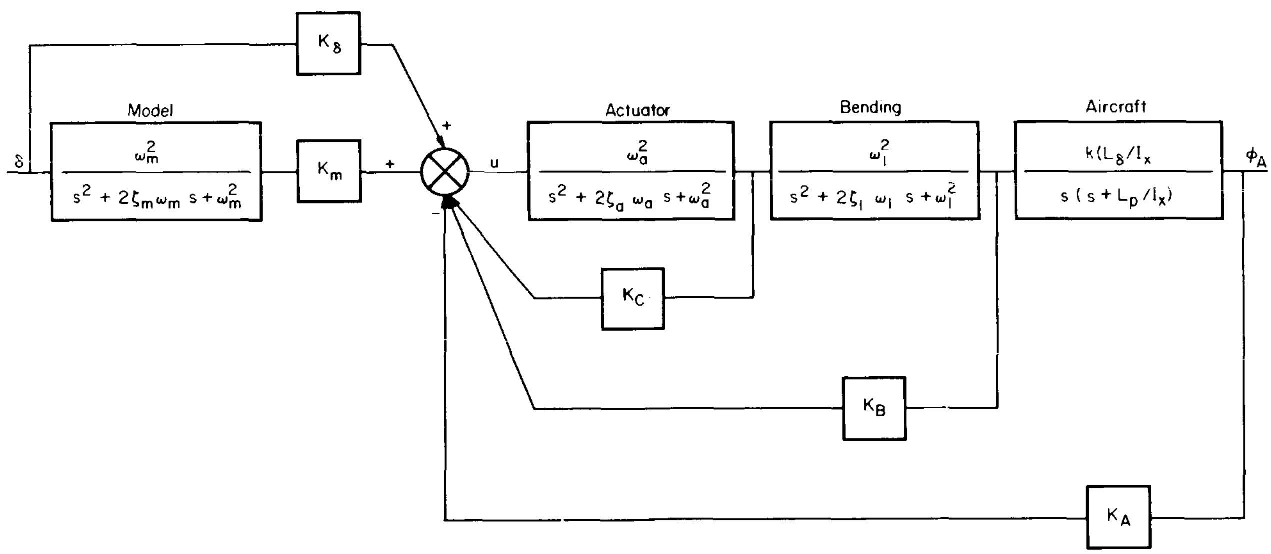

Figure 15.– Sixth-order follower block diagram.

There are several possible mathematical formulations of the system shown in figure 15. Hence the development of the particular representation utilized in the text for the fourth-order example cited above is given below. Consider this fourth-order follower system as configured in figure 16.

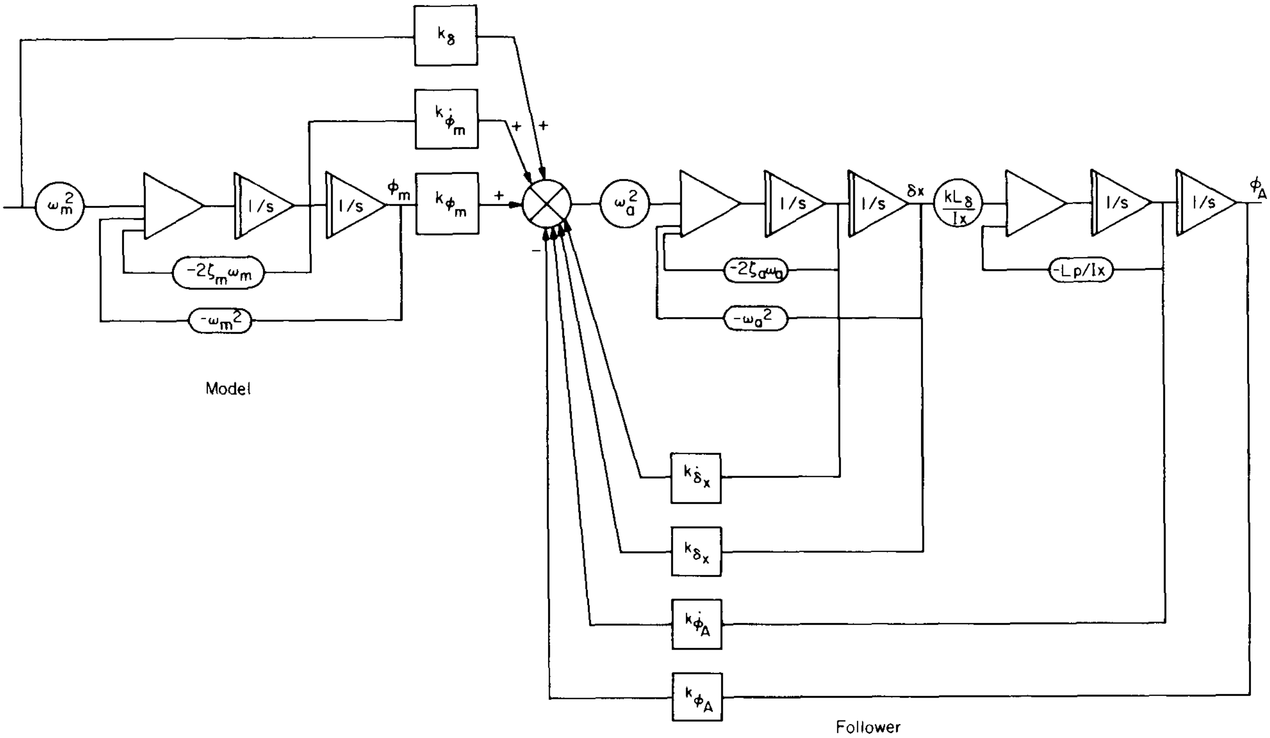

Figure 16.– Detailed system diagram.

The corresponding system equations are:

Model Equation

$$s^2\phi_m = -2\zeta_m\omega_m s\phi_m - \omega_m{}^2\phi_m + \omega_m{}^2\delta \tag{B1}$$

Follower Equations

$$\omega_a{}^2 u = (s^2 + 2\zeta_a\omega_a s + \omega_a{}^2)\delta_x \tag{B2}$$

$$k(L_\delta/I_x)\delta_x = s[s + (L_p/I_x)]\phi_A \tag{B3}$$

Control Equation

$$u = -(k_{\dot\delta_x}s + k_{\delta_x})\delta_x - (k_{\dot\phi_A}s + k_{\phi_A})\phi_A + \phi_c \tag{B4}$$

where

$$\phi_c = (k_{\dot\phi_m}s + k_{\phi_m})\phi_m + k_\delta\delta$$

An equivalent configuration representing the follower in its canonical form can be generated by first combining equations (B3) and (B4), that is,

$$u = \left[-\frac{k_{\dot{\delta}_x} I_x}{L_\delta} s^3 - \left(\frac{k_{\delta_x} I_x}{L_\delta} + \frac{k_{\dot{\delta}_x} L_p}{L_\delta} \right) s^2 - \left(\frac{k_{\delta_x} L_p}{L_\delta} + k_{\dot{\phi}_A} \right) s - k_{\phi_A} \right] \phi_A + \phi_c$$

$$= (-k_3 s^3 - k_2 s^2 - k_1 s - k_0) \phi_A + \phi_c \tag{B5}$$

Then combining equations (B2) and (B3) yields a composite follower equation from which the equivalent or representative system can be sketched:

$$k \frac{L_\delta}{I_x} \omega_a{}^2 u = \left[s^4 + \left(2\zeta_a \omega_a + \frac{L_p}{I_x} \right) s^3 + \left(\omega_a{}^2 + \frac{L_p}{I_x} 2\zeta_a \omega_a \right) s^2 + \omega_a{}^2 \frac{L_p}{I_x} s \right] \phi_A$$

$$= \left(s^4 - f_{44} s^3 \qquad - f_{43} s^2 \qquad - f_{42} s \right) \phi_A \tag{B6}$$

from which evolves figure 17.

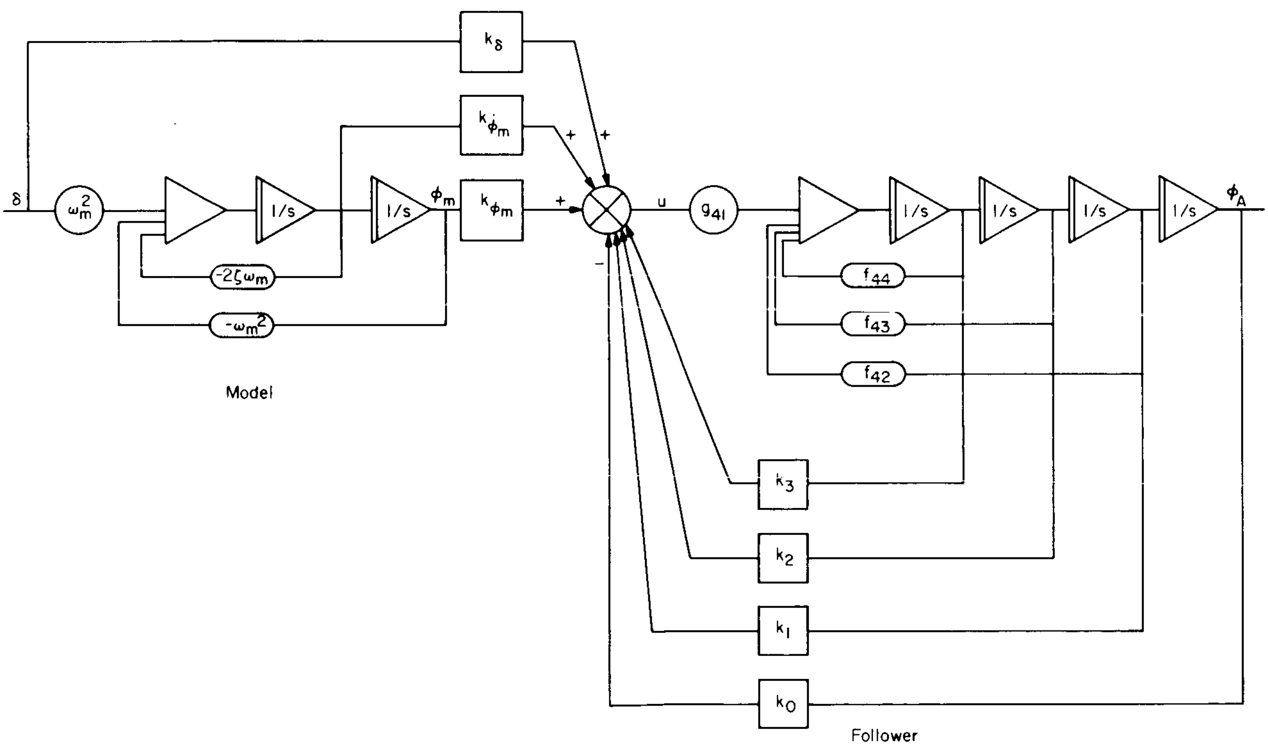

Figure 17.– Representative system diagram.

This figure represents a special case (i.e., $f_{41} = 0$) of the general representation shown on figure 3 in the text. The corresponding equations for figure 17 are

Model Equation

$$s^2\phi_m = -2\zeta_m\omega_m s\phi_m - \omega_m{}^2\phi_m + \omega_m{}^2\delta$$

Follower Equation

$$g_{41}u = [s^4 - f_{44}s^3 - f_{43}s^2 - f_{42}s]\phi_A$$

where

$$g_{41} = k(L_\delta/I_x)\omega_a{}^2 \tag{B7}$$

Control Equation

$$u = [-k_3s^3 - k_2s^2 - k_1s - k_0]\phi_A + \phi_c \tag{B8}$$

From these equations and again for δ equal to a step input, the state equation is

$$\frac{d}{dt}\begin{bmatrix}\phi_A\\ \dot\phi_A\\ \ddot\phi_A\\ \dddot\phi_A\\ \phi_m\\ \dot\phi_m\\ \delta\end{bmatrix} = \begin{bmatrix}0&1&0&0&0&0&0\\0&0&1&0&0&0&0\\0&0&0&1&0&0&0\\0&f_{42}&f_{43}&f_{44}&0&0&0\\0&0&0&0&0&1&0\\0&0&0&0&-\omega_m{}^2&-2\zeta_m\omega_m&\omega_m{}^2\\0&0&0&0&0&0&0\end{bmatrix}\begin{bmatrix}\phi_A\\ \dot\phi_A\\ \ddot\phi_A\\ \dddot\phi_A\\ \phi_m\\ \dot\phi_m\\ \delta\end{bmatrix} + \begin{bmatrix}0\\0\\0\\g_{41}\\0\\0\\0\end{bmatrix}u \tag{B9}$$

where

$$g_{41} = k(L_\delta/I_x)\omega_a{}^2$$

It is worth noting that equation (B9) is a special example of equation (5) in the text. For this representation, the total outer-loop gain D_1 referred to in the text is

$$D_1 = g_{41}(q_{11}/r)^{1/2} = k(L_\delta/I_x)\omega_a{}^2(q_{11}/r)^{1/2}$$

The nonhomogeneous state equation (eq. (B9)) can be programmed directly onto the analog computer. However, for digital computation, it is highly desirable to operate with a set of homogeneous equations. Under certain conditions (when the input can be represented by a state equation), such a set of equations can be generated from the nonhomogeneous form; a discussion of this procedure follows.

Homogeneous State Equation

$$\dot X = \bar F X \quad X_o = X(0) \tag{B10}$$

For systems in which the control equation can be expressed entirely in terms of the system states, that is,

$$u = KX$$

the homogeneous state equation can be written as

$$\underbrace{\frac{d}{dt}\begin{bmatrix} X' \\ \delta \end{bmatrix}}_{\dot{X}} = \underbrace{\left[\left[\begin{array}{c|c} F' & B \\ \hline 0 & E \end{array}\right] + [GK]\right]}_{\overline{F}} \underbrace{\begin{bmatrix} X' \\ \delta \end{bmatrix}}_{X} \tag{B11}$$

The model following systems considered in this study comply with this condition. With the fourth-order follower example shown in figure 18 for illustration, the homogeneous state equation of the form given by equation (B10) becomes .

$$\frac{d}{dt}\begin{bmatrix} \phi_A \\ \dot{\phi}_A \\ \ddot{\phi}_A \\ \dddot{\phi}_A \\ \phi_m \\ \dot{\phi}_m \\ \delta \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -gk_0 & (f_{42}-gk_1) & (f_{43}-gk_2) & (f_{44}-gk_3) & k_{\phi m} & k_{\dot{\phi} m} & k_\delta \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -\omega_m^2 & 2\zeta_m\omega_m & \omega_m^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \phi_A \\ \dot{\phi}_A \\ \ddot{\phi}_A \\ \dddot{\phi}_A \\ \phi_m \\ \dot{\phi}_m \\ \delta \end{bmatrix} \tag{B12}$$

The homogeneous equation for the second- and sixth-order examples may be generated in a similar manner and will not be shown here.

There are two factors worth noting in the preceding development. First, the natural or basic outer-loop follower feedback term is equal to zero (i.e., $f_{41} = 0$). Thus the transfer function for this follower is categorized as a non-type zero. The second factor is the distinction between the system matrix F of equation (B9) and $\overline{F}$ of equation (B11). F consists of the basic unaugmented follower and model matrices F_f and M, divorced from the augmenting feedback GK; that is,

$$F = \left[\begin{array}{cc|c} F_f & 0 & 0 \\ 0 & M & B \\ \hline 0 & 0 & E \end{array}\right] \tag{B13}$$

On the other hand, $\bar{F}$ represents a composite system matrix incorporating the augmenting feedback gains GK; that is,

$$\bar{F} = F + GK \tag{B14}$$

Hence, in the text, F and F_f are referred to as the unaugmented system matrix and the unaugmented follower matrix, respectively, and $\bar{F}$ is referred to as the augmented system matrix.

The goal achieved by the preceding development was the illustration of the modeling techniques used in this study and further, for ease of solution, the formation of a homogeneous state equation from a nonhomogeneous form.

Quadratic Criterion

The quadratic performance criterion, as given below, forms the basis of much modern and optimal control theory. Its minimization is achieved through a variety of techniques, including Hamilton-Jacobi, calculus of variations, or Pontryagin's maximum principle, most of which evolved into the solution of a first-order matrix differential equation referred to as the matrix Riccati equation. For a concise development of the Riccati equation, the reader is referred to references 5, 6, and 9. The Riccati equation, whose solution provides the designer with a set of system gains that will minimize the quadratic criterion, is given as

$$\dot{P} = PGR^{-1}G^TP - PF - F^TP - H^TQH \tag{B15}$$

A formulation and usage of this optimal technique will now be developed.

The model reference system under investigation is the prefilter type. To delineate the notation for the optimization of the regulator problem (i.e., no driving functions, $\delta = 0$), consider again the second-order model and linearized single-axis second-order aircraft follower configuration:

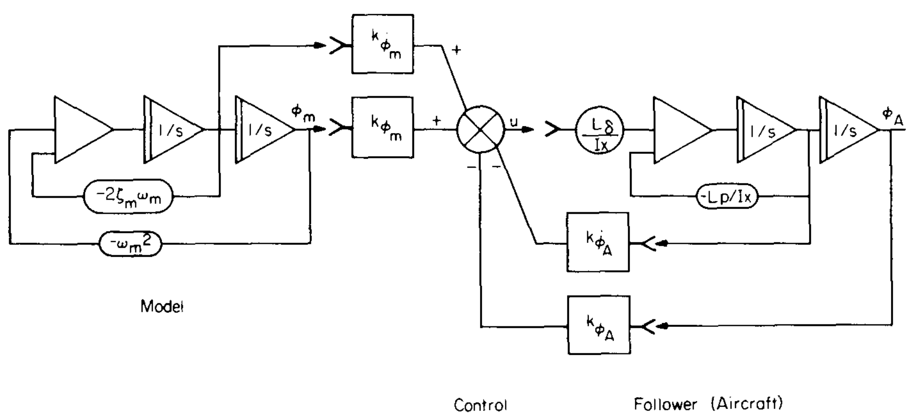

Figure 18.– Second-order follower system.

Statement of problem— Find a control u through a selection of gains K (i.e., $k_{\dot{\phi}_m}$, k_{ϕ_m}, $k_{\dot{\phi}_a}$, k_{ϕ_a}) such that the quadratic performance index is minimized.

$$V = \int_0^\infty X^T H^T Q H X + U^T R U \, dt \qquad \underline{\text{QPI}} \tag{B16}$$

Solution to the problem— Given the matrices F, G, H, Q, and R, which will be defined subsequently, the specific gains, K*, which minimize the QPI, can be determined through the solution of the matrix Riccati equation by the expression

$$K^* = -R^{-1} G^T P \tag{B17}$$

where P is the solution to the Riccati equation (eq. (B15)). The optimum control becomes

$$U^* = -R^{-1} G^T P X \qquad \underline{\text{Optimum Control}}$$

To define the notation used in this optimization procedure, consider the modeling of the second-order follower system shown above.

$$\frac{d}{dt}\begin{bmatrix} \phi_m \\ \dot{\phi}_m \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_m^2 & -2\zeta_m\omega_m \end{bmatrix}\begin{bmatrix} \phi_m \\ \dot{\phi}_m \end{bmatrix} \qquad \underline{\text{Model Equation}}$$

$$\dot{X}_m \quad = \quad M \quad X_m$$

$$u = \begin{bmatrix} k_{\phi_A} & k_{\dot{\phi}_A} & k_{\phi_m} & k_{\dot{\phi}_m} \end{bmatrix}\begin{bmatrix} \phi_A \\ \dot{\phi}_A \\ \phi_m \\ \dot{\phi}_m \end{bmatrix} \qquad \underline{\text{Control Equation}}$$

$$u = KX$$

$$\frac{d}{dt}\begin{bmatrix} \phi_A \\ \dot{\phi}_A \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -L_p/I_x \end{bmatrix}\begin{bmatrix} \phi_A \\ \dot{\phi}_A \end{bmatrix} \qquad \underline{\text{Follower Equation}}$$

$$\dot{X}_f \quad = \quad F_f \quad X_f$$

The key step in the optimization of this system is the determination of a matrix of gains K, which minimize the QPI. These gains can be determined through equation (B17), which requires first the solution of the Riccati equation in order to establish P. The Riccati equation is rather unruly; however, it can be handled nicely on a digital computer utilizing existing subroutines of the Automatic Synthesis Program (ASP) or by a more recently developed Fortran IV version of ASP, referred to as FASP (ref. 16). Required as inputs to these subroutines are the matrices F, G, H, Q, and R, which for the examples of figure 18 are shown below. The system matrix F is formulated as

$$\begin{bmatrix} \dot{X}_f \\ \dot{X}_m \end{bmatrix} = \left[\begin{bmatrix} F_f & 0 \\ 0 & M \end{bmatrix} + \begin{bmatrix} G_f K \\ 0 \end{bmatrix} \right] \begin{bmatrix} X_f \\ X_m \end{bmatrix}$$

System Equation

where the gains K in our case were initialized at zero. The other matrices are

$$X = \begin{bmatrix} X_f \\ X_m \end{bmatrix} = \begin{bmatrix} \phi_A \\ \dot{\phi}_A \\ \phi_m \\ \dot{\phi}_m \end{bmatrix} \qquad H = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$G = \begin{bmatrix} G_f \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ L_\delta/I_x \\ 0 \\ 0 \end{bmatrix} \qquad Q = \begin{bmatrix} q_{11} & 0 \\ 0 & q_{22} \end{bmatrix} \qquad R = r_{11}$$

Matrices H, Q, and R, all of which appear in the QPI integrand, are arbitrary matrices selected by the designer; H is constructed so as to include in the integrand the errors which are to be minimized. In the case above, the differences between the model and aircraft attitudes and rates are to be minimized; that is,

$$HX = \begin{bmatrix} \phi_m - \phi_A \\ \dot{\phi}_m - \dot{\phi}_A \end{bmatrix}$$

Matrices Q and R are the selected weighting matrices on the errors and on the cost of control. For this example, the scalar q_{11} is the weighting on the error of the attitude match, q_{22} on the rate match, and r_{11} on the control u.

This optimization technique, by its nature, requires that all states be available for feedforward or feedback. Such is, of course, not always the case; however, in many instances a near optimum can be achieved when a partial set of states is available. An additional feature of this procedure is that a different set of gains result from each combination of H, Q, and R. Hence there is a need to better understand the effects of changing these matrices on the optimums they produce.

Once the above system matrices have been generated as inputs for the programs ASP or FASP, the steady-state Riccati matrix P and the optimal feedback gain matrix K* can be determined by equation (B17). This optimal gain matrix can then be used to construct a model reference system that is "optimized" for the imposed set of constraints. The P matrix from which the K matrix is derived also contains much information regarding the ability of the optimized system to regulate about a variety of initial conditions or upsets. This information evolves from the expression

$$V_{min} = X_o^T P X_o \tag{B18}$$

where

$$X_o = X(0)$$

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